Solitary Wave Solution of a Benjamin-Bona-Mahony Equation by Using a Projective Riccati Equation Method

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Abstract - We apply a projective Riccati equation method to solve a Benjamin-Bona-Mahony equation. The equation describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. By using the method, we confirm the existence of solitary wave solution of the equation, which is expressed in a more general form.

Keywords - Benjamin-Bona-Mahony Equation, Projective Riccati Equation Method, Solitary Wave Solution.

I. INTRODUCTION

Nonlinear wave phenomena can be found in many problems of fluid dynamics, chemical reaction kinematics, cell and molecular biological processes, optical physics, etc. In many cases, the phenomena can be mathematically modeled by nonlinear partial differential equations (PDEs). The increasing attention on modeling using nonlinear PDEs leads to the development of various alternative methods to solve the equations. Some of the popular methods are tanh method, inverse scattering method, Adomian decomposition method and projective Riccati equation method [1].

In this paper, we particularly focus on the projective Riccati equation method. The method was firstly proposed by Conte and Mussete to find solitary wave solutions of some nonlinear PDEs [2]. They showed that the solitary wave solutions can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation. The Conte and Mussete’s method was further developed by Yan by introducing the general form of the projective Riccati equation [3]. Several authors have applied Yan’s technique to solve many nonlinear partial differential equations. For example, Li and Chen used the projective equation method to solve the nonlinear reaction-diffusion equation and the modified Bossiness equation [4]. They also applied the method to solve the general Korteweg-de Vries (KdV) type equation and the KdV-Burgers type equation [5]. In addition, the projective Riccati equation method is also implemented by Zhen and Hong-Qing to solve two nonlinear difference-differential equations, i.e the discrete Lotka-Voltera equation and the discrete Korteweg-de Vries (KdV) equation [6].

One of the nonlinear PDEs which take many attention of research is a Benjamin-Bona-Mahony (BBM) equation, which is written as [7]

\[ U_t + U_x + UU_x - U_{xxt} = 0. \]  

(1)

The BBM equation (1) was firstly investigated by Benjamin, Bona and Mahony as a rational alternative to the KdV equation for modeling the unidirectional propagation of small-amplitude long waves on the surface of water in a channel [8]. It is known that the BBM equation admits solitary wave solution which is given by [7]

\[ U(x,t) = \text{sech}^2 \left( \frac{x - t}{4} \right). \]  

(2)
Solitary wave solution of Eq. (1) has been widely studied by many authors using different methods, such as the exp-function method by Yusufoglu and Bekir [9,10], sinc-Galerkin approximation and tanh method by Alquran and Al-Khaled [11], and the homotopy perturbation method by Tari and Ganji [12]. Here we propose the projective Riccati equation method to obtain the solitary wave solution of the BBM equation (1).

The remainder of this paper is organized as follows. In Sec. 2, we review the projective Riccati equation method in a united scheme based on [13]. In Sec. 3, we apply the method to the BBM equation and confirm the existence of solitary wave solution. Finally, we provide conclusion in Sec. 4.

II. REVIEW OF THE PROJECTIVE RICCATI EQUATION METHOD

Consider a nonlinear partial differential equation (PDE)

\[ N(U, U_x, U_t, U_{xx}, \ldots) = 0. \]  

(3)

To obtain the traveling wave solution of equation (3), use the following wave transformation

\[ U(x,t) = u(\xi), \quad \xi = \omega x + ct, \]  

(4)

This in turn reduces Eq. (3) into a nonlinear ordinary differential equation

\[ M(u, u', u'', \ldots) = 0. \]  

(5)

A united scheme of projective Riccati equation method is given in the following steps [13]:

Step 1. Suppose the solution of (5) can be expressed as

\[ u(\xi) = H_1(f) + H_2(f), \]  

(6)

where

\[ H_1(f) = \sum_{i=0}^{m_1} b_i f^i, \quad H_2(f) = \sum_{i=0}^{m_2} c_i f^i. \]  

(7)

Here \( f = f(\xi) \) is a solution of the projective Riccati equation

\[ f' = fg, \]  

(8)

\[ g^2 = F(f) = \sum_{i=0}^{m_3} a_i f^i, \quad g' = \frac{1}{2} F'(f) f', \]  

(9)

Where \( m_1, m_2, m_3 \) are positive integers. After applying the chain rule \( \circ \) Eq. (6) and upon substituting Eqs. (8) and (9), we obtain the following equations:

Step 2. Applying the balance principle to obtain some relations for \( m_1 \), \( m_2 \) and \( m_3 \) from which the different possible values of \( m_1 \), \( m_2 \) and \( m_3 \) can be determined.

Step 3. By collecting the coefficients of all terms in \( G_1(f) \) and \( G_2(f) \) in power of \( f \) and setting each to be zeros, we will have a system of algebraic equations from which the values of \( a_i (i = 0, 1, 2, \ldots, m_1) \), \( b_i (i = 0, 1, 2, \ldots, m_2) \), and \( c_i (i = 0, 1, 2, \ldots, m_3) \) can be determined. One should note that the resulting system may not have solution, which means in this case Eq. (3) can not
be solved by the projective Riccati equation method for \( m_1, m_2 \) and \( m_3 \) resulted in Step 2.

**Step 4.** By squaring Eq. (8) and making use of Eq. (9), we will have.

\[
(f')^2 = (fg)^2 = f^2 F(f).
\]

Eq. (13) can be reduced to the elementary integral form as follows:

\[
\xi - \xi_0 = \pm \int \frac{df}{f\sqrt{F(f)}}.
\]

**Step 5.** By substituting the obtained values of \( a_i (i = 0, 1, 2, \ldots, m_3) \) into Eq. (14), and classifying the roots of \( F(f) \), we solve Eq. (14) and then obtain the exact solutions to Eq. (13). As the result, we can have the exact solutions to Eq. (3).

### III. Application of the Method to the BBM Equation

Recall the BBM equation as given by Eq. (1). By using traveling wave transformation (4), Eq. (1) becomes

\[
-c\omega^2 u'' + \alpha u' = -cu - \alpha uu'.
\]

Integrating Eq. (15) once, we obtain:

\[
-c\omega^2 u'' + \alpha u' = D - cu - \frac{\alpha}{2} u^2.
\]

For solitary wave’s solution, we impose the following boundary conditions:

\[
\begin{align*}
  u &\to 0 \text{ as } \xi \to \pm\infty, \\
  u' &\to 0 \text{ as } \xi \to \pm\infty, \\
  u'' &\to 0 \text{ as } \xi \to \pm\infty,
\end{align*}
\]

which in turn yields \( D = 0 \). Thus, Eq. (16) becomes

\[
-c\omega^2 u'' + \alpha u' = -cu - \frac{\omega}{2} u^2.
\]

Dividing Eq. (17) by \( c\omega^2 \) gives

\[
u'' = \frac{\omega + c}{c\omega^2} u + \frac{1}{2c\omega} u^2.
\]

By following Step 1 as explained in Sec. 2, we obtain

\[
G_1(f) + G_2(f) = 0,
\]

Where:

\[
G_1(f) = H_1(f) \omega f + \frac{1}{2} H_1''(f) f^2 F(f) - \frac{1}{2c\omega} |H_1(f)|^2 - \frac{1}{2c\omega} F(f) H_1(f) H_2(f) - \omega \frac{c}{c\omega} H_1(f)
\]

\[
G_2(f) = \frac{1}{2} F''(f) H_2(f) f^2 + \frac{3}{2} F'(f) H_2'(f) f^2 + \frac{3}{2} F'(f) H_2'(f) f^2 + \frac{3}{2} F'(f) H_2'(f) f^2 + H_2'(f) F(f) f^2
\]

\[
+ F(f) H_2'(f) f - \frac{1}{c\omega} H_1(f) H_2(f) - \frac{\alpha + c}{c\omega} H_2(f).
\]

As in Step 2, we have \( m_1 = m_3 \) and \( m_2 = m_1 / 2 \) after applying the balance principle to Eq. (19). For particular case, we choose \( m_1 = 1, m_2 = 0 \) and \( m_3 = 1 \). Thus, from Eqs. (7) and (9), we have

\[
H_1(f) = b_1 f + b_0,
\]

\[
H_2(f) = c_0,
\]

\[
F(f) = a_1 f + a_0.
\]

By substituting Eqs. (22), (23) and (24) into Eq. (19), and then, according to Step 3, setting the coefficients of all powers of \( f \) in \( G_1(f) \) and \( G_2(f) \) to zeroes, we arrive at the following system

\[
3 b_1 a_1 - \frac{1}{2c\omega} b_1^2 = 0,
\]

\[
3 b_1 a_1 - \frac{1}{2c\omega} b_1^2 = 0.
\]
\[ b_1 c_0 \rho_0 - \frac{1}{c \omega} b_1 c_0 - \frac{(\omega + c)}{c \omega^2} b_1 - \frac{1}{c \omega} b_1 c_0^2 = 0, \]  
\[ \frac{1}{2} a_0 c_0^2 - \frac{1}{c c_0^2} = 0, \]  
\[ \frac{1}{2 c \omega} c_0^2 + \frac{1}{2 c \omega} c_0^2 + \frac{(\omega + c)}{c \omega^2} b_0 + \frac{1}{c \omega} b_0 c_0 + \frac{3}{2} b_1 a_1 + \frac{(\omega + c)}{c \omega^2} - c_0 = 0. \]

Solutions of the above system are as follows:

(i) \[ a_0 = \frac{b_0^2 \omega + 2 b_0 c_0 \omega + 2 b_0 c + 2 b_0 \omega + 2 c c_0 + 2 b_0 \omega}{c_0^2 \omega}, \quad a_1 = 0, \quad b_0 = b_0, \quad b_1 = 0, \]
\[ c = c, \quad c_0 = c_0, \quad \omega = \omega, \]

(ii) \[ a_0 = a_0, \quad a_1 = a_1, \quad b_0 = 0, \quad b_0 = 0, \quad c = 0, \quad c_0 = 0, \quad \omega = \omega, \]

(iii) \[ a_0 = \frac{(\omega + c)}{c \omega^2}, \quad a_1 = \frac{b_1}{3 c \omega}, \quad b_0 = 0, \quad c = c, \quad c_0 = 0, \quad \omega = \omega, \]

(iv) \[ a_0 = a_0, \quad a_1 = a_1, \quad b_0 = -\frac{(\omega + c)}{\omega}, \quad b_1 = 0, \quad c = c, \quad c_0 = 0, \quad \omega = \omega, \]

(v) \[ a_0 = -\frac{(\omega + c)}{c \omega^2}, \quad a_1 = a_1, \quad b_0 = \frac{b_1}{3 c \omega}, \quad b_1 = b_1, \quad c = c, \quad c_0 = 0, \quad \omega = \omega, \]

(vi) \[ a_0 = \frac{b_0^2 \omega^2 + 2 b_0 c_0 \omega^2 + 2 b_0 \omega^2 + 2 c c_0 \omega^2 + 2 b_0 c_0 + 2 b_0 \omega}{c_0^2 \omega^2}, \quad a_1 = a_1, \quad b_0 = b_0, \quad b_1 = 0, \]
\[ c = \frac{c_0}{c_0}, \quad c_0 = c_0, \quad \omega = \omega. \]

Next, as in Step 4, solving the integration (14), where \( F(f) \) is given by Eq. (24), yield
\[ \sqrt{a_0} (\xi - \xi_0) = \mp 2 \text{ arc tanh} \left( \frac{\sqrt{a_0 f + a_0}}{a_0} \right), \]
\[ \text{which then gives} \]
\[ f = -\frac{a_0 \text{sech}^2 \left( \frac{1}{2} \sqrt{a_0 (\xi - \xi_0)} \right)}{a_1}. \]

After substituting Eqs. (22), (23) and (24) into Eq. (6), where \( f \) is given by Eq. (36), we have solution to Eq. (18) as follows:
\[ u(\xi) = -\frac{a_0 b_1 \text{sech}^2 \left( \frac{1}{2} \sqrt{a_0 (\xi - \xi_0)} \right)}{a_1} + b_0 \pm c_0 \sqrt{a_0 \text{sech}^2 \left( \frac{1}{2} \sqrt{a_0 (\xi - \xi_0)} \right)} + a_0. \]

If we use (31), then solution (37) with relation (4) becomes
which is solitary wave solution of BBM equation (1) in a more general form. One can check that if \(c = -4, \omega = 3, \xi_0 = 0\) are chosen, then the resulting solution is exactly the same as given by Eq. (2).

IV. CONCLUSION

The projective Riccati equation method has been applied in this paper to find the exact solitary wave solution of the Benjamin-Bona-Mahoney equation. We obtained that the resulting solution is expressed in a more general form than that found previously.

REFERENCES


