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Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Sets and its Application on Multi-Attribute Decision-Making by Using Aggregation Operator Method

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Abstract—**Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Sets (IVBNHFS) can express complex multi-attribute decision-making (MADM) problems with its interval valued, bipolar, neutrosophic, and hesitant fuzzy elements simultaneously. The two weighted operators, average and geometric, are accustomed aggregation operator method for handling MADM problems. However, we give the Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Average (IVBNHFWA) and Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Geometric (IVBNHFWG) operators to aggregate the IVBNHFS and their properties are also discussed in detail. Occasionally, we predestined a MADM method based on the IVBNHFWA and IVBNHFWG operators. Finally, this paper provides an illustrative numerical example of a case study of MADM to authenticate the application and effectiveness of the proposed approaches. Also, we give a comparative study with the existing methods.**

Keywords—Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Set (IVBNHFS); Weighted Average (WA) operator; Weighted Geometric (WG) operator, Multi-Attribute Decision-Making (MADM), Score functions, Decision making.

I. INTRODUCTION

The aim of this paper is to develop a new method for Multi-Attribute Decision-Making (MADM) under an Interval Valued Bipolar Neutrosophic Hesitant Fuzzy (IVBNHF) environment. The MADM is a common approach [1-7] to help decision makers to find the most desirable alternative from some possible choices.

To deal with MADM, Fuzzy Set (FS) [8] is given officially by Zadeh in 1965. On basis of Zadeh's work, Intuitionistic Fuzzy Set (IFS) [9], Neutrosophic set (NS) [10], Bipolar Fuzzy Sets (BFS) [11], Hesitant Fuzzy Set (HFS) [12], Hesitant Neutrosophic Sets (HNS) [13], have been proposed and applied in handling with various uncertain problems in decision making. Therefore, many researchers studied decision making with neutrosophic theory in many fields: multi-attribute decision-making (MADM) [1-7], multi-criteria decision-making (MCDM) [14-21], and other types of decision making [22-34]. In the practical decision-making application, it is important to use various techniques to aggregate information [35-36]. The one of the effective techniques to aggregate data in the information fusion process is the aggregation operators [5], [17], [25], [35]. Recently, Ramaroson and al. [37] adopted Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Sets (IVBNHFS) [37]. Therefore, it is necessary to develop some Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Sets operators in MADM problem. Thus, the purposes of this article are to (i) introduce the Weighted Average (WA) and Geometric Average (WG) operators into IVBNHFS in MADM, (ii) propose an Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Average (IVBNHFWA) operator and an Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Geometric (IVBNHFWG) operator (iii) establish MADM strategies based on the IVBNHFWA or IVBNHFWG operators under IVBNHF environment.

So, the remainder of this paper is coordinated as follows: Section II reviews some essential ideas related to the Interval Valued Bipolar Neutrosophic Sets (IVBNHFS) ; In Section III, the IVBNHFWA and IVBNHFWG operators are put forward and basic properties of them are discussed; In Section IV, we put forward MADM methods based on the weighted operators (IVBNHFWA and IVBNHFWG) under IVBNHF information; Section V uses an illustrative example to approve the proposed MADM approaches. In Section VI, we make a comparative analysis with the Deli's method [38]. At last in Section VI, conclusion and future perspectives are presented.

II. Mathematical Preliminaries

In this section, some basic concepts (properties, operation law, theorem and example) of Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Sets (IVBNHFS) are recalled.

Definition 2.1. (IVBNHFS) [37] Assume X is a nonempty finite set, an IVBNHFS P on X is defined as:

$$P = \{\langle x, t^+(x), i^+(x), f^+(x), t^-(x), i^-(x), f^-(x)\rangle | \ x \in X\}$$

$$P = \{\langle x, t^+(x), i^+(x), f^+(x), t^-(x), i^-(x), f^-(x)\rangle | \ x \in X\} \tag{1}$$

Where $t^+(x) = \{\gamma^+ | \gamma^+ \in t^+(x)\}, i^+(x) = \{\delta^+ | \delta^+ \in i^+(x)\}$, $f^+(x) = \{\eta^+ | \eta^+ \in f^+(x)\}$ are positive three membership functions expressed by a few closed intervals in the real unit interval [0,1] which represent the truth positive membership hesitant degree, indeterminacy positive membership hesitant degree and falsity positive membership hesitant degree and meet the following conditions:

$$\gamma^+ = [\gamma_L^+, \gamma_U^+] \in [0,1], \delta^+ = [\delta_L^+, \delta_U^+] \in [0,1], \eta^+ = [\eta_L^+, \eta_U^+] \in [0,1]$$
$$0 \leq sup\gamma^{m+} + sup\delta^{m+} + sup\eta^{m+} \leq 3$$

where $\gamma^{m+} = \bigcup_{\gamma^+ \in t^+(x)} \max \gamma^+$, $\delta^{m+} = \bigcup_{\delta^+ \in i^+(x)} \max \delta^+$ and $\eta^{m+} = \bigcup_{\eta^+ \in f^+(x)} \max \eta^+$.

And $t^-(x) = \{\gamma^- | \gamma^- \in t^-(x)\}, i^-(x) = \delta^- | \delta^- \in i^-(x)$, $f^-(x) = \{\eta^- | \eta^- \in f^-(x)\}$ are negative three membership functions expressed by a few closed intervals in the real unit interval $[-1,0]$ which represent the truth negative membership hesitant degree, indeterminacy negative membership hesitant degree and falsity negative membership hesitant degree and meet the following conditions:

$$\gamma^- = [\gamma_L^-, \gamma_U^-] \in [-1, 0], \delta^- = [\delta_L^-, \delta_U^-] \in [-1, 0], \eta^- = [\eta_L^-, \eta_U^-] \in [-1,0]$$
$$-3 \leq sup\gamma^{m-} + sup\delta^{m-} + sup\eta^{m-} \leq 0$$

where $\gamma^{m-} = \bigcup_{\gamma^- \in t^-(x)} \max \gamma^-$, $\delta^{m-} = \bigcup_{\delta^- \in i^-(x)} \max \delta^-$ and $\eta^{m-} = \bigcup_{\eta^- \in f^-(x)} \max \eta^-$.

Example 2.2.

Let $X = \{x_1, x_2, x_3, x_4\}$. Then:

$$P = \left\{ \begin{array}{c} \langle x_1, \langle \begin{array}{c} \{\{[0.5, 0.6]\}, \{[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]\}, \{[0.1, 0.2], [0.2, 0.3], [0.3, 0.4], [0.4, 0.5], [0.5, 0.6]\}, \\ \{[-0.2, -0.1]\}, \{[-0.6, -0.5], [-0.5, -0.4], [-0.4, -0.3], [-0.3, -0.2]\}, \{[-0.4, -0.3]\}\} \end{array} \rangle\rangle \\ \langle x_2, \langle \begin{array}{c} \{\{[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8], [0.8,0.9]\}, \{[0.1,0.2], [0.2,0.3], [0.3,0.4], [0.4,0.5]\} \\ , \{[0.2,0.3], [0.3,0.4], [0.4,0.5]\}, \{[-0.8, -0.7]\}, \{[-0.5, -0.4], [-0.4, -0.3], [-0.3, -0.2], [-0.2, -0.1]\}, \\ \{[-0.4, -0.3], [-0.3, -0.2], [-0.2, -0.1]\}\} \end{array} \rangle\rangle \\ \langle x_3, \langle \begin{array}{c} \{\{[0.1, 0.2], [0.2, 0.3], [0.3, 0.4], [0.4, 0.5], [0.5, 0.6]\}, \{[0.1, 0.2], [0.2, 0.3], [0.3, 0.4], [0.4, 0.5]\}, \{[0.1, 0.2]\} \\ , \{[-0.5, -0.4], [-0.4, -0.3], [-0.3, -0.2]\}, \{[-0.7, -0.6], [-0.6, -0.5], [-0.5, -0.4], [-0.4, -0.3]\}, \\ \{[-0.4, -0.3], [-0.3, -0.2]\}\} \end{array} \rangle\rangle \\ \langle x_4, \langle \begin{array}{c} \{\{[0.2,0.3], [0.3,0.4], [0.4,0.5], [0.5,0.6]\}, \{[0.1,0.2], [0.2,0.3], [0.3,0.4], [0.4,0.5]\}, \{[0.3,0.4]\}, \\ \{[-0.4, -0.3], [-0.3, -0.2]\}, \{[-0.7, -0.6], [-0.6, -0.5], [-0.5, -0.4], [-0.4, -0.3]\}, \{ [-0.3, -0.2]\}\} \end{array} \rangle\rangle \end{array} \right\}$$

is an IVBNHFS in X.

Theorem 2.3.

An IVBNHFS is a generalization of Interval Bipolar Neutrosophic Fuzzy Sets (IBNFS).

Proof.

Assume that the number of elements in each γ^+, δ^+, $\eta^+, \gamma^-, \delta^-$, and η^- is one, then IVBNHFS is reduce to IBNFS. □

Definition 2.4. (IVBNHFS Properties) [37] Let $A = \langle [t_{LA}^+, t_{UA}^+], [i_{LA}^+, i_{UA}^+], [f_{LA}^+, f_{UA}^+], [t_{LA}^-, t_{UA}^-], [i_{LA}^-, i_{UA}^-], [f_{LA}^-, f_{UA}^-]\rangle$, and $B = \langle [t_{LB}^+, t_{UB}^+], [i_{LB}^+, i_{UB}^+], [f_{LB}^+, f_{UB}^+], [t_{LB}^-, t_{UB}^-], [i_{LB}^-, i_{UB}^-], [f_{LB}^-, f_{UB}^-]\rangle$; be two IVBNHFS:

- the complement of an IVBNHFS A, A^C and is defined by:

$$A^C = \bigcup_{\substack{\gamma_A^+ \in t_A^+, \ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-, \ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \left\{ \langle \begin{array}{c} [1 - \gamma_{LA}^+(x), 1 - \gamma_{UA}^+(x)], \\ [1 - \delta_{LA}^+(x), 1 - \delta_{UA}^+(x)], \\ [1 - \eta_{LA}^+(x), 1 - \eta_{UA}^+(x)], \\ [-1 - \gamma_{LA}^-(x), 1 - \gamma_{UA}^-(x)], \\ [-1 - \delta_{LA}^-(x), -1 - \delta_{UA}^-(x)], \\ [-1 - \eta_{LA}^-(x), -1 - \eta_{UA}^-(x)] \end{array} \rangle \right\} \tag{2}$$

- the intersection of a two IVBNHFS A and B, $A \cap B$ is defined by:

$$A \cap B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\wedge(\gamma_{LA}^+, \gamma_{LB}^+), \wedge(\gamma_{UA}^+, \gamma_{UB}^+)], \\ [\vee(\delta_{LA}^+, \delta_{LB}^+), \vee(\delta_{UA}^+, \delta_{UB}^+)], \\ [\vee(\eta_{LA}^+, \eta_{LB}^+), \vee(\eta_{UA}^+, \eta_{UB}^+)], \\ [\wedge(\gamma_{LA}^-, \gamma_{LB}^-), \wedge(\gamma_{UA}^-, \gamma_{UB}^-)], \\ [\vee(\delta_{LA}^-, \delta_{LB}^-), \vee(\delta_{UA}^-, \delta_{UB}^-)], \\ [\vee(\eta_{LA}^-, \eta_{LB}^-), \vee(\eta_{UA}^-, \eta_{UB}^-)] \end{matrix} \right\} \quad (3)$$

- the union of a two IVBNHFS A and B, $A \cup B$ is defined by:

$$A \cup B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\vee(\gamma_{LA}^+, \gamma_{LB}^+), \vee(\gamma_{UA}^+, \gamma_{UB}^+)], \\ [\wedge(\delta_{LA}^+, \delta_{LB}^+), \wedge(\delta_{UA}^+, \delta_{UB}^+)], \\ [\wedge(\eta_{LA}^+, \eta_{LB}^+), \wedge(\eta_{UA}^+, \eta_{UB}^+)], \\ [\vee(\gamma_{LA}^-, \gamma_{LB}^-), \vee(\gamma_{UA}^-, \gamma_{UB}^-)], \\ [\wedge(\delta_{LA}^-, \delta_{LB}^-), \wedge(\delta_{UA}^-, \delta_{UB}^-)], \\ [\wedge(\eta_{LA}^-, \eta_{LB}^-), \wedge(\eta_{UA}^-, \eta_{UB}^-)] \end{matrix} \right\} \quad (4)$$

Example 2.5.
Let A and B

$$A = \left\langle \begin{matrix} \{\{[0.5, 0.6]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[-0.2, -0.1]\}, \\ \{[-0.6, -0.5], [-0.5, -0.4]\}, \\ \{[-0.4, -0.3]\}\} \end{matrix} \right\rangle$$

$$B = \left\langle \begin{matrix} \{\{[0.4, 0.5]\}, \\ \{[0.3, 0.4], [0.4, 0.5]\}, \\ \{[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]\}, \\ \{[-0.1, 0]\}, \\ \{[-0.5, -0.4]\}, \\ \{[-0.5, -0.4]\}\} \end{matrix} \right\rangle$$

are two IVBNHFEs, then:

- the complement of A and B are:

$$A^C = \left\langle \begin{matrix} \{\{[0.5, 0.4]\}, \\ \{[0.8, 0.7], [0.7, 0.6]\}, \\ \{[0.9, 0.8], [0.8, 0.7]\}, \\ \{[-0.8, -0.9]\}, \\ \{[-0.4, -0.5], [-0.5, -0.6]\}, \\ \{[-0.6, -0.7]\}\} \end{matrix} \right\rangle$$

$$B^C = \left\langle \begin{matrix} \{\{[0.6, 0.5]\}, \\ \{[0.7, 0.6], [0.6, 0.5]\}, \\ \{[0.8, 0.7], [0.7, 0.6], [0.6, 0.5]\}, \\ \{[-0.9, -1]\}, \\ \{[-0.5, -0.6]\}, \\ \{[-0.5, -0.6]\}\} \end{matrix} \right\rangle$$

- the intersection of A and B, $A \cap B$ is:

$$A \cap B = \left\langle \begin{matrix} \{\{[0.4, 0.5]\}, \\ \{[0.3, 0.4], [0.4, 0.5]\}, \\ \{[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]\}, \\ \{[-0.2, -0.1]\}, \\ \{[-0.5, -0.4]\}, \\ \{[-0.4, -0.3]\}\} \end{matrix} \right\rangle$$

- the union of A and B, $A \cup B$ is:

$$A \cup B = \left\langle \begin{matrix} \{\{[0.5, 0.6]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[-0.1, -0]\}, \\ \{[-0.6, -0.5], [-0.5, -0.4]\}, \\ \{[-0.5, -0.4]\}\} \end{matrix} \right\rangle$$

Definition 2.6. (IVBNHFS Operations) [37]
Let $A = \langle [t^+_{LA}, t^+_{UA}], [i^+_{LA}, i^+_{UA}], [f^+_{LA}, f^+_{UA}], [t^-_{LA}, t^-_{UA}], [i^-_{LA}, i^-_{UA}], [f^-_{LA}, f^-_{UA}] \rangle$, and
$B = \langle [t^+_{LB}, t^+_{UB}], [i^+_{LB}, i^+_{UB}], [f^+_{LB}, f^+_{UB}], [t^-_{LB}, t^-_{UB}], [i^-_{LB}, i^-_{UB}], [f^-_{LB}, f^-_{UB}] \rangle$; be two IVBNHFS:

- the sum algebraic is defined as follows:

$$A \oplus B = \bigcup_{\substack{\gamma^+_A \in t^+_A,\ \delta^+_A \in i^+_A, \eta^+_A \in f^+_A, \\ \gamma^-_A \in t^-_A,\ \delta^-_A \in i^-_A, \eta^-_A \in f^-_A, \\ \gamma^+_B \in t^+_B,\ \delta^+_B \in i^+_B, \eta^+_B \in f^+_B, \\ \gamma^-_B \in t^-_B,\ \delta^-_B \in i^-_B, \eta^-_B \in f^-_B}} \left\{ \begin{matrix} [\gamma^+_{LA} + \gamma^+_{LB} - \gamma^+_{LA}\gamma^+_{LB}, \gamma^+_{UA} + \gamma^+_{UB} - \gamma^+_{UA}\gamma^+_{UB}], \\ [\delta^+_{LA}\delta^+_{LB}, \delta^+_{UA}\delta^+_{UB}], \\ [\eta^+_{LA}\eta^+_{LB}, \eta^+_{UA}\eta^+_{UB}], \\ [-\gamma^-_{LA}\gamma^-_{LB}, -\gamma^-_{UA}\gamma^-_{UB}], \\ [-(-\delta^-_{LA} - \delta^-_{LB} - \delta^-_{LA}\delta^-_{LB}), -(-\delta^-_{UA} - \delta^-_{UB} - \delta^-_{UA}\delta^-_{UB})], \\ [-(-\eta^-_{LA} - \eta^-_{LB} - \eta^-_{LA}\eta^-_{LB}), -(-\eta^-_{UA} - \eta^-_{UB} - \eta^-_{UA}\eta^-_{UB})], \end{matrix} \right\} \quad (5)$$

- the product is defined as follows:

$$A \otimes B = \bigcup_{\substack{\gamma^+_A \in t^+_A,\ \delta^+_A \in i^+_A, \eta^+_A \in f^+_A, \\ \gamma^-_A \in t^-_A,\ \delta^-_A \in i^-_A, \eta^-_A \in f^-_A, \\ \gamma^+_B \in t^+_B,\ \delta^+_B \in i^+_B, \eta^+_B \in f^+_B, \\ \gamma^-_B \in t^-_B,\ \delta^-_B \in i^-_B, \eta^-_B \in f^-_B}} \left\{ \begin{matrix} [\gamma^+_{LA}\gamma^+_{LB}, \gamma^+_{UA}\gamma^+_{UB}], \\ [\delta^+_{LA} + \delta^+_{LB} - \delta^+_{LA}\delta^+_{LB}, \delta^+_{UA} + \delta^+_{UB} - \delta^+_{UA}\delta^+_{UB}], \\ [\eta^+_{LA} + \eta^+_{LB} - \eta^+_{LA}\eta^+_{LB}, \eta^+_{UA} + \eta^+_{UB} - \eta^+_{UA}\eta^+_{UB}], \\ [-(-\gamma^-_{LA} - \gamma^-_{LB} - \gamma^-_{LA}\gamma^-_{LB}), -(-\gamma^-_{UA} - \gamma^-_{UB} - \gamma^-_{UA}\gamma^-_{UB})], \\ [-\delta^-_{LA}\delta^-_{LB}, -\delta^-_{UA}\delta^-_{UB}], \\ [-\eta^-_{LA}\eta^-_{LB}, -\eta^-_{UA}\eta^-_{UB}] \end{matrix} \right\} \quad (6)$$

- when λ > 0, the scalar multiplication of A is λA, and is defined as follows:

$$\lambda A = \bigcup_{\substack{\gamma^+_A \in t^+_A,\ \delta^+_A \in i^+_A, \eta^+_A \in f^+_A, \\ \gamma^-_A \in t^-_A,\ \delta^-_A \in i^-_A, \eta^-_A \in f^-_A}} \left\{ \begin{matrix} [1 - (1 - \gamma^+_{LA})^\lambda, 1 - (1 - \gamma^+_{UA})^\lambda], \\ [(\delta^+_{LA})^\lambda, (\delta^+_{UA})^\lambda], \\ [(\eta^+_{LA})^\lambda, (\eta^+_{UA})^\lambda], \\ [-(-\gamma^-_{LA})^\lambda, -(-\gamma^-_{UA})^\lambda], \\ \left[-\left(1 - (1 - (-\delta^-_{LA}))^\lambda\right), -\left(1 - (1 - (-\delta^-_{LA}))^\lambda\right)\right], \\ \left[-\left(1 - (1 - (-\eta^-_{LA}))^\lambda\right), -\left(1 - (1 - (-\eta^-_{LA}))^\lambda\right)\right] \end{matrix} \right\} \quad (7)$$

- when λ > 0, the power multiplication of A is A^λ, and is defined as follows:

$$A^\lambda = \bigcup_{\substack{\gamma^+_A \in t^+_A,\ \delta^+_A \in i^+_A, \eta^+_A \in f^+_A, \\ \gamma^-_A \in t^-_A,\ \delta^-_A \in i^-_A, \eta^-_A \in f^-_A}} \left\{ \begin{matrix} [(\gamma^+_{LA})^\lambda, (\gamma^+_{UA})^\lambda] \\ [1 - (1 - \delta^+_{LA})^\lambda, 1 - (1 - \delta^+_{UA})^\lambda], \\ [1 - (1 - \eta^+_{LA})^\lambda, 1 - (1 - \eta^+_{UA})^\lambda], \\ \left[-\left(1 - (1 - (-\gamma^-_{LA}))^\lambda\right), -\left(1 - (1 - (-\gamma^-_{LA}))^\lambda\right)\right], \\ [-(-\delta^-_{LA})^\lambda, -(-\delta^-_{UA})^\lambda] \\ [-(-\eta^-_{LA})^\lambda, -(-\eta^-_{UA})^\lambda] \end{matrix} \right\} \quad (8)$$

Example 2.7.

Let A and B

$$A = \langle \begin{matrix} \{\{[0.5, 0.6]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[-0.2, -0.1]\}, \\ \{[-0.6, -0.5]\}, \\ \{[-0.4, -0.3]\}\} \end{matrix} \rangle$$

$$B = \langle \begin{matrix} \{\{[0.4, 0.5]\}, \\ \{[0.3, 0.4], [0.4, 0.5]\}, \\ \{[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]\}, \\ \{[-0.1, 0]\}, \\ \{[-0.5, -0.4]\}, \\ \{[-0.5, -0.4]\}\} \end{matrix} \rangle$$

are two IVBNHFEs, then:

- the sum algebraic is defined as follows:

$$A \oplus B = \langle \begin{matrix} \{\{[0.7, 0.8]\}, \\ \{[0.06, 0.12], [0.08, 0.15], [0.09, 0.16], [0.12, 0.2]\}, \\ \{[0.02, 0.06], [0.03, 0.08], [0.04, 0.1], [0.04, 0.09], [0.06, 0.12], [0.08, 0.15]\}, \\ \{[-0.02, 0]\}, \\ \{[-0.8, -0.7]\}, \\ \{[-0.7, -0.58]\}\} \end{matrix} \rangle$$

- the product is defined as follows:

$$A \otimes B = \langle \begin{matrix} \{\{[0.2, 0.3]\}, \\ \{[0.44, 0.58], [0.5, 0.7], [0.5, 0.6], [0.6, 0.7]\}, \\ \{[0.28, 0.44], [0.4, 0.5], [0.5, 0.6], [0.36, 0.51], [0.4, 0.6], [0.5, 0.7]\}, \\ \{[-0.3, -0.1]\}, \\ \{[-0.3, -0.2]\}, \\ \{[-0.2, -0.12]\}\} \end{matrix} \rangle$$

- when $\lambda = 0.5$, the scalar multiplication of A is $0.5A$, and is defined as follows:

$$0.5A = \langle \begin{matrix} \{\{[0.293, 0.225]\}, \\ \{[0.447, 0.548], [0.548, 0.632]\}, \\ \{[0.316, 0.447], [0.447, 0.548]\}, \\ \{[-0.2, -0.1]\}, \\ \{[-0.632, -0.707]\}, \\ \{[-0.775, -0.837]\}\} \end{matrix} \rangle$$

- when $\lambda = 0.5$, the power multiplication of A is $A^{0.5}$, and is defined as follows:

$$A^{0.5} = \langle \begin{matrix} \{\{[0.707, 0.775]\}, \\ \{[0.106, 0.163], [0.163, 0.225]\}, \\ \{[0.053, 0.106], [0.106, 0.163]\}, \\ \{[-0.106, -0.051]\}, \\ \{[-0.775, -0.71]\}, \\ \{[-0.632, -0.55]\}\} \end{matrix} \rangle$$

III. PROPOSED AGGREGATION OPERATOR METHOD FOR IVBNHFS

We propose two weighted operators for IVBNHFS.

Definition 3.1. (IVBNHFWA Operator)

Let $A_j = \langle [t^+_{LAj}, t^+_{UAj}], [i^+_{LAj}, i^+_{UAj}], [f^+_{LAj}, f^+_{UAj}], [t^-_{LAj}, t^-_{UAj}], [i^-_{LAj}, i^-_{UAj}], [f^-_{LAj}, f^-_{UAj}] \rangle$, be a family of Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Elements ($IVBNHFEs$).

A mapping $IVBNHFWA_\omega$: $\widehat{H}_n \longrightarrow \widehat{H}$ is called Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Average ($IVBNHFWA$) operator if it satisfies:

$$IVBNHFWA_\omega(A_{2HZS1}, A_{2HZS2}, \ldots, A_{2HZSn}) = \sum_{j=1}^{n} \omega_j A_{2HZSj} \tag{9}$$

$$= \langle \bigcup_{\gamma^+_{Lj} \in t^+_{Lj}, \gamma^+_{Uj} \in t^+_{Uj}} \left\{ 1 - \prod_{j=1}^{n} \left(1 - \gamma^+_{Lj}\right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \gamma^+_{Uj}\right)^{\omega_j} \right\},$$

$$\bigcup_{\delta^+_{Lj} \in i^+_{Lj},\ \delta^+_{Uj} \in i^+_{Uj}} \left\{ \prod_{j=1}^{n} \left(\delta^+_{Lj}\right)^{\omega_j}, \prod_{j=1}^{n} \left(\delta^+_{Uj}\right)^{\omega_j} \right\}, \bigcup_{\eta^+_{Lj} \in f^+_{Lj},\ \eta^+_{Uj} \in f^+_{Uj}} \left\{ \prod_{j=1}^{n} \left(\eta^+_{Lj}\right)^{\omega_j}, \prod_{j=1}^{n} \left(\eta^+_{Uj}\right)^{\omega_j} \right\},$$

$$\bigcup_{\gamma^-_{Lj} \in t^-_{Lj}, \gamma^-_{Uj} \in t^-_{Uj}} \left\{ -\prod_{j=1}^{n} \left(-\gamma^-_{Lj}\right)^{\omega_j}, -\prod_{j=1}^{n} \left(-\gamma^-_{Uj}\right)^{\omega_j} \right\},$$

$$\bigcup_{\delta^-_{Lj} \in i^-_{Lj},\ \delta^-_{Uj} \in i^-_{Uj}} \left\{ -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\delta^-_{Lj}\right)\right)^{\omega_j}, -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\delta^-_{Uj}\right)\right)^{\omega_j}, \right)\right) \right\},$$

$$\bigcup_{\eta^-_{Lj} \in f^-_{Lj},\ \eta^-_{Uj} \in f^-_{Uj}} \left\{ -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\eta^-_{Lj}\right)\right)^{\omega_j} \right), -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\eta^-_{Uj}\right)\right)^{\omega_j} \right) \right\} \rangle$$

Where ω_j is the weight of $A_j(A_1, A_2, \ldots, A_n)$, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Then $IVBNHFWA_\omega(A_1, A_2, \ldots, A_n)$ is called $IVBNHFWA$ and the result of the aggregation are still $IVBNHFEs$.

Theorem 3.2.

Let $A_j = \langle [t^+_{LAj}, t^+_{UAj}], [i^+_{LAj}, i^+_{UAj}], [f^+_{LAj}, f^+_{UAj}], [t^-_{LAj}, t^-_{UAj}], [i^-_{LAj}, i^-_{UAj}], [f^-_{LAj}, f^-_{UAj}] \rangle$, be a family of $IVBNHFEs$. Then,

i) Idempotency

If $A_j = A$ for all $j = 1, 2, \ldots, n$ then

$$IVBNHFWA_\omega(A_1, A_2, \ldots, A_n) = A$$

ii) Monotonicity

If $A_j \leq A^*_j$ for all $j = 1, 2, \ldots, n$ then

$$IVBNHFWA_\omega(A_1, A_2, \ldots, A_n) \leq IVBNHFWA^*_\omega(A^*_1, A^*_2, \ldots, A^*_n)$$

iii) Boundedness

$$\min_{j=1,2,\ldots,n}\{A_j\} \leq IVBNHFWA_\omega(A_1, A_2, \ldots, A_n) \leq \max_{j=1,2,\ldots,n}\{A_j\}$$

Proof. i) Idempotency

Since $A_j = A = \langle [t^+_{LA}, t^+_{UA}], [i^+_{LA}, i^+_{UA}], [f^+_{LA}, f^+_{UA}], [t^-_{LA}, t^-_{UA}], [i^-_{LA}, i^-_{UA}], [f^-_{LA}, f^-_{UA}] \rangle$, for all j, we have:

$$IVBNHFWA_\omega\left(A_{2HZSj}\right) = \sum_{j=1}^{n} \omega_j A_j$$

$$= \langle \bigcup_{\gamma^+_L \in t^+_L, \gamma^+_U \in t^+_U} \left\{ 1 - \prod_{j=1}^{n} (1 - \gamma^+_L)^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \gamma^+_U)^{\omega_j} \right\}, \bigcup_{\delta^+_L \in i^+_L,\ \delta^+_U \in i^+_U} \left\{ \prod_{j=1}^{n} (\delta^+_L)^{\omega_j}, \prod_{j=1}^{n} (\delta^+_U)^{\omega_j} \right\},$$

$$\bigcup_{\eta^+_L \in f^+_L,\ \eta^+_U \in f^+_U} \left\{ \prod_{j=1}^{n} (\eta^+_L)^{\omega_j}, \prod_{j=1}^{n} (\eta^+_U)^{\omega_j} \right\}, \bigcup_{\gamma^-_L \in t^-_L, \gamma^-_U \in t^-_U} \left\{ -\prod_{j=1}^{n} (-\gamma^-_L)^{\omega_j}, -\prod_{j=1}^{n} (-\gamma^-_U)^{\omega_j} \right\},$$

$$\bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_L^-)\right)^{\omega_j},-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_U^-)\right)^{\omega_j},\right)\right)\right\},$$

$$\bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\eta_L^-)\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\eta_U^-)\right)^{\omega_j}\right)\right\}\rangle$$

$$=\langle \bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\left\{1-(1-\gamma_L^+)^{\sum_{j=1}^{n}\omega_j},1-(1-\gamma_U^+)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\left\{(\delta_L^+)^{\sum_{j=1}^{n}\omega_j},(\delta_U^+)^{\sum_{j=1}^{n}\omega_j}\right\},$$

$$\bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\left\{(\eta_L^+)^{\sum_{j=1}^{n}\omega_j},(\eta_U^+)^{\sum_{j=1}^{n}\omega_j}\right\}, \bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\left\{(-\gamma_L^-)^{\sum_{j=1}^{n}\omega_j},-(-\gamma_U^-)^{\sum_{j=1}^{n}\omega_j}\right\},$$

$$\bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\left\{-\left(1-\left(1-(-\delta_L^-)\right)^{\sum_{j=1}^{n}\omega_j},-\left(1-\left(1-(-\delta_U^-)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right)\right\},$$

$$\bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\left\{-\left(1-\left(1-(-\eta_L^-)\right)^{\sum_{j=1}^{n}\omega_j}\right),-\left(1-\left(1-(-\eta_U^-)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\}\rangle$$

Since $\sum_{j=1}^{n}\omega_j = 1$, we have:

$$=\langle \bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\{1-(1-\gamma_L^+),1-(1-\gamma_U^+)\}, \bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\{(\delta_L^+),(\delta_U^+)\}, \bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\{(\eta_L^+),(\eta_U^+)\},$$

$$\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\{-(-\gamma_L^-),-(-\gamma_U^-)\}, \bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\left\{-\left(1-\left(1-(-\delta_L^-)\right),-\left(1-\left(1-(-\delta_U^-)\right)\right)\right)\right\},$$

$$\bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\left\{-\left(1-\left(1-(-\eta_L^-)\right)\right),-\left(1-\left(1-(-\eta_U^-)\right)\right)\right\}\rangle$$

$$=\langle \bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\{(\gamma_L^+),(\gamma_U^+)\}, \bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\{(\delta_L^+),(\delta_U^+)\}, \bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\{(\eta_L^+),(\eta_U^+)\},$$

$$\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\{(\gamma_L^-),(\gamma_U^-)\}, \bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\{(\delta_L^-),(\delta_U^-)\}, \bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\{(\eta_L^-),(\eta_U^-)\}\rangle$$

$$\langle[t_{LA}^+,t_{UA}^+],[i_{LA}^+,i_{UA}^+],[f_{LA}^+,f_{UA}^+],[t_{LA}^-,t_{UA}^-],[i_{LA}^-,i_{UA}^-],[f_{LA}^-,f_{UA}^-]\rangle = A.$$

Which completes the proof of i)

ii) Monotonicity

Since $t_{LAj}^+ \le {t_{LAj}^+}^*$, and $t_{UAj}^+ \le {t_{UAj}^+}^*$,for all j, then we have

$\gamma_{Lj}^+ \le {\gamma_{Lj}^+}^*, 1-\gamma_{Lj}^+ \ge 1-{\gamma_{Lj}^+}^*$ and $\gamma_{Uj}^+ \le {\gamma_{Uj}^+}^*, 1-\gamma_{Uj}^+ \ge 1-{\gamma_{Uj}^+}^*$

$$\left\{\prod_{j=1}^{n}\left(1-\gamma_{Lj}^+\right)^{\omega_j},\prod_{j=1}^{n}\left(1-\gamma_{Uj}^+\right)^{\omega_j}\right\} \ge \left\{\prod_{j=1}^{n}\left(1-{\gamma_{Lj}^+}^*\right)^{\omega_j},\prod_{j=1}^{n}\left(1-{\gamma_{Uj}^+}^*\right)^{\omega_j}\right\}$$

$$\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{Lj}^+\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-\gamma_{Uj}^+\right)^{\omega_j}\right\} \le \left\{1-\prod_{j=1}^{n}\left(1-{\gamma_{Lj}^+}^*\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-\gamma_{Uj}^+\right)^{\omega_j}\right\}$$

$$\left[\bigcup_{\gamma_{Lj}^+\in t_{LAj}^+}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{Lj}^+\right)^{\omega_j}\right\}, \bigcup_{\gamma_{Uj}^+\in t_{UAj}^+}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{Uj}^+\right)^{\omega_j}\right\}\right]$$

$$\le \left[\bigcup_{\gamma_{Lj}^+\in t_{LAj}^+}\left\{1-\prod_{j=1}^{n}\left(1-{\gamma_{Lj}^+}^*\right)^{\omega_j}\right\}, \bigcup_{\gamma_{Uj}^+\in t_{UAj}^+}\left\{1-\prod_{j=1}^{n}\left(1-{\gamma_{Uj}^+}^*\right)^{\omega_j}\right\}\right]$$

Since $i_{LAj}^+ \le {i_{LAj}^+}^*$and $i_{UAj}^+ \le {i_{UAj}^+}^*$for all j, then we have $\delta_{Lj}^+ \le {\delta_{Lj}^+}^*$and $\delta_{Uj}^+ \le {\delta_{Lj}^+}^*$

$$\left\{\prod_{j=1}^{n}\left(\delta_{Lj}^+\right)^{\omega_j},\prod_{j=1}^{n}\left(\delta_{Uj}^+\right)^{\omega_j}\right\} \le \left\{\prod_{j=1}^{n}\left({\delta_{Lj}^+}^*\right)^{\omega_j},\prod_{j=1}^{n}\left({\delta_{Uj}^+}^*\right)^{\omega_j}\right\}$$

$$\left[\left\{\bigcup_{\delta_{Lj}^{+}\in i_{LAj}^{+}}\left\{\prod_{j=1}^{n}(\delta_{Lj}^{+})^{\omega_j}\right\}\right\},\left\{\bigcup_{\delta_{Uj}^{+}\in i_{UAj}^{+}}\left\{\prod_{j=1}^{n}(\delta_{Uj}^{+})^{\omega_j}\right\}\right\}\right]$$
$$\leq\left\{\bigcup_{\delta_{Lj}^{+}\in i_{LAj}^{+}}\left\{\prod_{j=1}^{n}(\delta_{Lj}^{+*})^{\omega_j}\right\}\right\},\left\{\bigcup_{\delta_{Uj}^{+}\in i_{UAj}^{+}}\left\{\prod_{j=1}^{n}(\delta_{Uj}^{+*})^{\omega_j}\right\}\right\}$$

Since $f_{LAj}^{+}\geq f_{LAj}^{+*}$ and $f_{UAj}^{+}\geq f_{UAj}^{+*}$ for all j, then we have $\eta_{Lj}^{+}\geq\eta_{Lj}^{+*}$ and $\eta_{Uj}^{+}\geq\eta_{Uj}^{+*}$

$$\left[\prod_{j=1}^{n}(\eta_{Lj}^{+})^{\omega_j},\prod_{j=1}^{n}(\eta_{Uj}^{+})^{\omega_j}\right]\geq\left[\prod_{j=1}^{n}(\eta_{Lj}^{+*})^{\omega_j},\prod_{j=1}^{n}(\eta_{Uj}^{+*})^{\omega_j}\right]$$
$$\left[\left\{\bigcup_{\eta_{Lj}^{+}\in f_{LAj}^{+}}\left\{\prod_{j=1}^{n}(\eta_{Lj}^{+})^{\omega_j}\right\}\right\},\left\{\bigcup_{\eta_{Uj}^{+}\in f_{UAj}^{+}}\left\{\prod_{j=1}^{n}(\eta_{Uj}^{+})^{\omega_j}\right\}\right\}\right]$$
$$\geq\left[\left\{\bigcup_{\eta_{Lj}^{+}\in f_{LAj}^{+}}\left\{\prod_{j=1}^{n}(\eta_{Lj}^{+*})^{\omega_j}\right\}\right\},\left\{\bigcup_{\eta_{Uj}^{+}\in f_{UAj}^{+}}\left\{\prod_{j=1}^{n}(\eta_{Uj}^{+*})^{\omega_j}\right\}\right\}\right]$$

Since $t_{LAj}^{-}\geq t_{LAj}^{-*}$ and $t_{UAj}^{-}\geq t_{UAj}^{-*}$ for all j, then we have $\gamma_{Lj}^{-}\geq\gamma_{Lj}^{-*}$, $-\gamma_{Lj}^{-}\leq-\gamma_{Lj}^{-*}$ and $\gamma_{Uj}^{-}\geq\gamma_{Uj}^{-*}$, $-\gamma_{Uj}^{-}\leq-\gamma_{Uj}^{-*}$

$$\left[\prod_{j=1}^{n}(-\gamma_{Lj}^{-})^{\omega_j},\prod_{j=1}^{n}(-\gamma_{Uj}^{-})^{\omega_j}\right]\leq\left[\prod_{j=1}^{n}(-\gamma_{Lj}^{-*})^{\omega_j},\prod_{j=1}^{n}(-\gamma_{Uj}^{-*})^{\omega_j}\right]$$
$$\left[-\prod_{j=1}^{n}(-\gamma_{Lj}^{-})^{\omega_j},-\prod_{j=1}^{n}(-\gamma_{Uj}^{-})^{\omega_j}\right]\geq\left[-\prod_{j=1}^{n}(-\gamma_{Lj}^{-*})^{\omega_j},-\prod_{j=1}^{n}(-\gamma_{Uj}^{-*})^{\omega_j}\right]$$
$$\left[\left\{\bigcup_{\gamma_{Lj}^{-}\in t_{LAj}^{-}}\left\{-\prod_{j=1}^{n}(-\gamma_{Lj}^{-})^{\omega_j}\right\}\right\},\left\{\bigcup_{\gamma_{Uj}^{-}\in t_{UAj}^{-}}\left\{-\prod_{j=1}^{n}(-\gamma_{Uj}^{-})^{\omega_j}\right\}\right\}\right]$$
$$\geq\left[\left\{\bigcup_{\gamma_{Lj}^{-}\in t_{LAj}^{-}}\left\{-\prod_{j=1}^{n}(-\gamma_{Lj}^{-*})^{\omega_j}\right\}\right\},\left\{\bigcup_{\gamma_{Uj}^{-}\in t_{UAj}^{-}}\left\{-\prod_{j=1}^{n}(-\gamma_{Uj}^{-*})^{\omega_j}\right\}\right\}\right]$$

Since $i_{LAj}^{-}\geq i_{LAj}^{-*}$ and $i_{UAj}^{-}\geq i_{UAj}^{-*}$ for all j, then we have $\delta_{Lj}^{-}\geq\delta_{Lj}^{-*}$, $-\delta_{Lj}^{-}\leq-\delta_{Lj}^{-*}$ and $\delta_{Uj}^{-}\geq\delta_{Uj}^{-*}$, $-\delta_{Uj}^{-}\leq-\delta_{Uj}^{-*}$; $1-(-\delta_{Lj}^{-})\geq1-(-\delta_{Lj}^{-*})$ and $1-(-\delta_{Lj}^{-})\geq1-(-\delta_{Lj}^{-*})$

$$\left[\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-})\right)^{\omega_j},\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-})\right)^{\omega_j},\right]\geq\left[\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-*})\right)^{\omega_j},\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-*})\right)^{\omega_j}\right]$$
$$\left[1-\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-})\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-})\right)^{\omega_j}\right]$$
$$\leq\left\{1-\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-*})\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-*})\right)^{\omega_j}\right\}$$
$$\left[-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-})\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-})\right)^{\omega_j}\right)\right]$$
$$\geq\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-*})\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-*})\right)^{\omega_j}\right)\right\}$$
$$\left[\bigcup_{\delta_{Lj}^{-}\in i_{LAj}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Lj}^{-})\right)^{\omega_j}\right)\right\},\bigcup_{\delta_{Lj}^{-}\in i_{LAj}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\delta_{Uj}^{-})\right)^{\omega_j}\right)\right\}\right]$$

$$\geq \left[\bigcup_{\delta_{Lj}^- \in i_{LAj}^-} \left\{ -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\delta_{Lj}^{-*}\right)\right)^{\omega_j}\right) \right\}, \bigcup_{\delta_{Lj}^- \in i_{LAj}^-} \left\{ -\left(1 - \prod_{j=1}^{n} \left(1 - \left(-\delta_{Uj}^{-*}\right)\right)^{\omega_j}\right) \right\} \right]$$

Since $f_{LAj}^- \leq f_{LAj}^{-*}$ and $f_{LUj}^- \leq f_{LUj}^{-*}$ for all j, then we have $n_{L_j}^- \leq n_{L_j}^{-*}$, $-n_{L_j}^- \geq -n_{L_j}^{-*}$ and $n_{U_j}^- \leq n_{U_j}^{-*}$, $-n_{U_j}^- \geq -n_{U_j}^{-*}$; $1 - \left(-n_{L_j}^-\right) \leq 1 - \left(-n_{L_j}^{-*}\right)$ and $1 - \left(-n_{U_j}^-\right) \leq 1 - \left(-n_{U_j}^{-*}\right)$

$$\left[\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^-\right)\right)^{\omega_j}, \prod_{j=1}^{n}\left(1-\left(-n_{U_j}^-\right)\right)^{\omega_j}\right] \leq \left[\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^{-*}\right)\right)^{\omega_j}, \prod_{j=1}^{n}\left(1-\left(-n_{U_j}^{-*}\right)\right)^{\omega_j}\right]$$

$$\left[1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^-\right)\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\left(-n_{U_j}^-\right)\right)^{\omega_j}\right]$$

$$\geq \left[1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^{-*}\right)\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\left(-n_{U_j}^{-*}\right)\right)^{\omega_j}\right]$$

$$\left[-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^-\right)\right)^{\omega_j}\right), -\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{U_j}^-\right)\right)^{\omega_j}\right)\right]$$

$$\leq \left[-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^{-*}\right)\right)^{\omega_j}\right), -\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{U_j}^{-*}\right)\right)^{\omega_j}\right)\right]$$

$$\left[\bigcup_{n_{L_j}^- \in f_{LAj}^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^-\right)\right)^{\omega_j}\right)\right\}, \bigcup_{n_{U_j}^- \in f_{LAj}^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{U}^-\right)\right)^{\omega_j}\right)\right\}\right]$$

$$\leq \left[\bigcup_{n_{L_j}^- \in f_{LAj}^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{L_j}^{-*}\right)\right)^{\omega_j}\right)\right\}, \bigcup_{n_{U_j}^- \in f_{LAj}^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-n_{U_j}^{-*}\right)\right)^{\omega_j}\right)\right\}\right]$$

Based on above analysis, we have

$$IVBNHFWA_{\omega}(A_{2HZS1}, A_{2HZS2}, \dots, A_{2HZSn}) \leq IVBNHFWA_{\omega}^{*}(A_{2HZS1}^{*}, A_{2HZS2}^{*}, \dots, A_{2HZSn}^{*})$$

which completes the proof of the theorem 1 ii).

iii) Boundedness

Similar to the proof of ii), thus omitted. □

Definition 3.3. (IVBNHFWG Operator)

Let $A_j = \langle [t_{LAj}^+, t_{UAj}^+], [i_{LAj}^+, i_{UAj}^+], [f_{LAj}^+, f_{UAj}^+], [t_{LAj}^-, t_{UAj}^-], [i_{LAj}^-, i_{UAj}^-], [f_{LAj}^-, f_{UAj}^-] \rangle$, be a family of $IVBNHFEs$. A mapping $IVBNHFWG_{\omega}$: $\widehat{H}_n \longrightarrow \widehat{H}$ is called Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Geometric ($IVBNHFWG$) operator if it satisfies:

$$IVBNHFWG_{\omega}(A_1, A_2, \dots, A_n) = \sum_{j=1}^{n} \left(A_j\right)^{\omega_j} \tag{10}$$

$$= \langle \bigcup_{\gamma_{Lj}^+ \in t_{Lj}^+, \gamma_{Uj}^+ \in t_{Uj}^+} \left\{\prod_{j=1}^{n}\left(\gamma_{Lj}^+\right)^{\omega_j}, \prod_{j=1}^{n}\left(\gamma_{Uj}^+\right)^{\omega_j}\right\},$$

$$\bigcup_{\delta_{Lj}^+ \in i_{Lj}^+,\ \delta_{Uj}^+ \in i_{Uj}^+} \left\{1-\prod_{j=1}^{n}\left(1-\delta_{Lj}^+\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\delta_{Uj}^+\right)^{\omega_j}\right\},$$

$$\bigcup_{\eta_{Lj}^+ \in f_{Lj}^+,\ \eta_{Uj}^+ \in f_{Uj}^+} \left\{1-\prod_{j=1}^{n}\left(1-\eta_{Lj}^+\right)^{\omega_j}, 1-\prod_{j=1}^{n}\left(1-\eta_{Uj}^+\right)^{\omega_j}\right\},$$

$$\bigcup_{\gamma_{Lj}^- \in t_{Lj}^-, \gamma_{Uj}^- \in t_{Uj}^-} \left\{-\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{Lj}^-\right)\right)^{\omega_j}\right), -\left(1-\prod_{j=1}^{n}\left(1-\left(-\gamma_{Uj}^-\right)\right)^{\omega_j}\right)\right\},$$

$$\bigcup_{\delta_{Lj}^-\in i_{Lj}^-,\ \delta_{Uj}^-\in i_{Uj}^-}\left\{\left(-\prod_{j=1}^{n}\left(-\delta_{Lj}^-\right)^{\omega_j},-\prod_{j=1}^{n}\left(-\delta_{Uj}^-\right)^{\omega_j},\right)\right\},$$

$$\bigcup_{\eta_{Lj}^-\in f_{Lj}^-,\ \eta_{Uj}^-\in f_{Uj}^-}\left\{\left(-\prod_{j=1}^{n}\left(-\eta_{Lj}^-\right)^{\omega_j}\right),\left(-\prod_{j=1}^{n}\left(-\eta_{Uj}^-\right)^{\omega_j}\right)\right\}\rangle$$

Where ω_j is the weight of $A_{2HZSj}(A_{2HZS1},A_{2HZS2},\dots,A_{2HZSn})$, $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Then $IVBNHFWG_\omega(A_{2HZS1},A_{2HZS2},\dots,A_{2HZSn})$ is called $IVBNHFWG$ and the result of the aggregation are still $IVBNHFEs$.

Theorem 3.4.

Let $A_j=\langle[t_{LAj}^+,t_{UAj}^+],[i_{LAj}^+,i_{UAj}^+],[f_{LAj}^+,f_{UAj}^+],[t_{LAj}^-,t_{UAj}^-],[i_{LAj}^-,i_{UAj}^-],[f_{LAj}^-,f_{UAj}^-]\rangle$, be a family of $IVBNHFEs$. Then,

i) Idempotency

If $A_j=A$ for all $j=1,2,\dots,n$ then

$$IVBNHFWG_\omega(A_1,A_2,\dots,A_n)=A$$

ii) Monotonicity

If $A_j\le A_j^*$ for all $j=1,2,\dots,n$ then

$$IVBNHFWG_\omega(A_1,A_2,\dots,A_n)\le IVBNHFWG_\omega^*(A_1^*,A_2^*,\dots,A_n^*)$$

iii) Boundedness

$$\min_{j=1,2,\dots,n}\{A_j\}\le IVBNHFWG_\omega(A_1,A_2,\dots,A_n)\le\max_{j=1,2,\dots,n}\{A_j\}$$

Proof. i) Idempotency

Since $A_j=A=\langle[t_{LA}^+,t_{UA}^+],[i_{LA}^+,i_{UA}^+],[f_{LA}^+,f_{UA}^+],[t_{LA}^-,t_{UA}^-],[i_{LA}^-,i_{UA}^-],[f_{LA}^-,f_{UA}^-]\rangle$, for all j, we have:

$IVBNHFWG_\omega(A_j)=\sum_{j=1}^{n}(A_j)^{\omega_j}$

$$=\langle\bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\left\{\prod_{j=1}^{n}(\gamma_L^+)^{\omega_j},\prod_{j=1}^{n}(\gamma_U^+)^{\omega_j}\right\},\bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\left\{1-\prod_{j=1}^{n}(1-\delta_L^+)^{\omega_j},1-\prod_{j=1}^{n}(1-\delta_U^+)^{\omega_j}\right\},$$

$$\bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\left\{1-\prod_{j=1}^{n}(1-\eta_L^+)^{\omega_j},1-\prod_{j=1}^{n}(1-\eta_U^+)^{\omega_j}\right\},$$

$$\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_L^-)\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_U^-)\right)^{\omega_j}\right)\right\},$$

$$\bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\left\{\left(-\prod_{j=1}^{n}(-\delta_L^-)^{\omega_j},-\prod_{j=1}^{n}(-\delta_U^-)^{\omega_j},\right)\right\},\bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\left\{\left(-\prod_{j=1}^{n}(-\eta_L^-)^{\omega_j}\right),\left(-\prod_{j=1}^{n}(-\eta_U^-)^{\omega_j}\right)\right\}\rangle$$

$$=\langle\bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\left\{(\gamma_L^+)^{\sum_{j=1}^{n}\omega_j},(\gamma_U^+)^{\sum_{j=1}^{n}\omega_j}\right\},\bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\left\{1-(1-\delta_L^+)^{\sum_{j=1}^{n}\omega_j},1-(1-\delta_U^+)^{\sum_{j=1}^{n}\omega_j}\right\},$$

$$\bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\left\{1-(1-\eta_L^+)^{\sum_{j=1}^{n}\omega_j},1-(1-\eta_U^+)^{\sum_{j=1}^{n}\omega_j}\right\},$$

$$\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\left\{-\left(1-\left(1-(-\gamma_L^-)\right)^{\sum_{j=1}^{n}\omega_j}\right),-\left(1-\left(1-(-\gamma_U^-)\right)^{\sum_{j=1}^{n}\omega_j}\right)\right\},$$

$$\bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\left\{\left(-(-\delta_L^-)^{\sum_{j=1}^{n}\omega_j},-(-\delta_U^-)^{\sum_{j=1}^{n}\omega_j},\right)\right\},\bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\left\{\left(-(-\eta_L^-)^{\sum_{j=1}^{n}\omega_j}\right),\left(-(-\eta_U^-)^{\sum_{j=1}^{n}\omega_j}\right)\right\}\rangle$$

Since $\sum_{j=1}^{n}\omega_j=1$, we have

$$=\langle\bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\{(\gamma_L^+),(\gamma_U^+)\},\bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\{1-(1-\delta_L^+),1-(1-\delta_U^+)\},$$

$$\bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\{1-(1-\eta_L^+),1-(1-\eta_U^+)\},\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\left\{-\left(1-\left(1-(-\gamma_L^-)\right)\right),-\left(1-\left(1-(-\gamma_U^-)\right)\right)\right\},$$

$$\bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\{(-(-\delta_L^-),-(-\delta_U^-),)\},\ \bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\{(-(-\eta_L^-)),(-(-\eta_U^-))\}\rangle$$

$$=\langle\bigcup_{\gamma_L^+\in t_L^+,\gamma_U^+\in t_U^+}\{(\gamma_L^+),(\gamma_U^+)\},\ \bigcup_{\delta_L^+\in i_L^+,\ \delta_U^+\in i_U^+}\{(\delta_L^+),(\delta_U^+)\},\ \bigcup_{\eta_L^+\in f_L^+,\ \eta_U^+\in f_U^+}\{(\eta_L^+),(\eta_U^+)\},$$

$$\bigcup_{\gamma_L^-\in t_L^-,\gamma_U^-\in t_U^-}\{(\gamma_L^-),(\gamma_U^-)\},\ \bigcup_{\delta_L^-\in i_L^-,\ \delta_U^-\in i_U^-}\{(\delta_L^-),(\delta_U^-)\},\ \bigcup_{\eta_L^-\in f_L^-,\ \eta_U^-\in f_U^-}\{(\eta_L^-),(\eta_U^-)\}\rangle$$

$$\langle[t_{LA}^+,t_{UA}^+],[i_{LA}^+,i_{UA}^+],[f_{LA}^+,f_{UA}^+],[t_{LA}^-,t_{UA}^-],[i_{LA}^-,i_{UA}^-],[f_{LA}^-,f_{UA}^-]\rangle=A.$$

Which completes the proof of the theorem 1 i)

ii) Monotonicity

Since $t_{LAj}^+\le t_{LAj}^{+\,*}$, and $t_{UAj}^+\le t_{UAj}^{+\,*}$,for all j, then we have:

$\gamma_{Lj}^+\le\gamma_{Lj}^{+\,*}$and $\gamma_{Uj}^+\le\gamma_{Uj}^{+\,*}$.

$$\left[\prod_{j=1}^{n}(\gamma_{Lj}^+)^{\omega_j},\prod_{j=1}^{n}(\gamma_{Uj}^+)^{\omega_j}\right]\le\left[\prod_{j=1}^{n}(\gamma_{Lj}^{+\,*})^{\omega_j},\prod_{j=1}^{n}(\gamma_{Uj}^{+\,*})^{\omega_j}\right]$$

$$\left[\bigcup_{\gamma_{Lj}^+\in h_{T_j}^+}\left\{\prod_{j=1}^{n}(\gamma_{Lj}^+)^{\omega_j}\right\},\bigcup_{\gamma_{Uj}^+\in h_{T_j}^+}\left\{\prod_{j=1}^{n}(\gamma_{Uj}^+)^{\omega_j}\right\}\right]\le\left[\bigcup_{\gamma_{Lj}^+\in h_{T_j}^+}\left\{\prod_{j=1}^{n}(\gamma_{Lj}^{+\,*})^{\omega_j}\right\},\bigcup_{\gamma_{Uj}^+\in h_{T_j}^+}\left\{\prod_{j=1}^{n}(\gamma_{Uj}^{+\,*})^{\omega_j}\right\}\right]$$

Since $i_{LAj}^+\le i_{LAj}^{+\,*}$ and $i_{UAj}^+\le i_{UAj}^{+\,*}$ for all j, then we have $\delta_{Lj}^+\le\delta_{Lj}^{+\,*}$, $\delta_{Uj}^+\le\delta_{Uj}^{-\,*}$ $1-\delta_{Lj}^+\ge1-\delta_{Lj}^{+\,*}$ and $1-\delta_{Lj}^-\ge1-\delta_{Lj}^{-\,*}$

$$\left\{\prod_{j=1}^{n}(1-\delta_{Lj}^+)^{\omega_j}\right\}\ge\left\{\prod_{j=1}^{n}(1-\delta_{Lj}^{+\,*})^{\omega_j}\right\}$$

$$\left\{1-\prod_{j=1}^{n}(1-\delta_{Lj}^+)^{\omega_j}\right\}\le\left\{1-\prod_{j=1}^{n}(1-\delta_{Lj}^{+\,*})^{\omega_j}\right\}$$

$$\left\{\bigcup_{\gamma_{I_j}^+\in h_{I_j}^+}\left\{1-\prod_{j=1}^{n}(1-\delta_{Lj}^+)^{\omega_j}\right\}\right\}\le\left\{\bigcup_{\gamma_{I_j}^+\in h_{I_j}^+}\left\{1-\prod_{j=1}^{n}(1-\delta_{Lj}^{+\,*})^{\omega_j}\right\}\right\}$$

Since $f_{LAj}^+\ge f_{LAj}^{+\,*}$and $f_{UAj}^+\ge f_{UAj}^{+\,*}$ for all j, then we have $\eta_{Lj}^+\ge\eta_{Lj}^{+\,*}$, $\eta_{Uj}^+\ge\eta_{Uj}^{+\,*}$, $1-\eta_{Lj}^+\le1-\eta_{Lj}^{+\,*}$and $1-\eta_{Lj}^-\le1-\eta_{Lj}^{-\,*}$

$$\left[\prod_{j=1}^{n}(1-\eta_{Lj}^+)^{\omega_j},\prod_{j=1}^{n}(1-\eta_{Uj}^+)^{\omega_j}\right]\le\left[\prod_{j=1}^{n}(1-\eta_{Lj}^{+\,*})^{\omega_j},\prod_{j=1}^{n}(1-\eta_{Uj}^{+\,*})^{\omega_j}\right]$$

$$\left[1-\prod_{j=1}^{n}(1-\eta_{Lj}^+)^{\omega_j},1-\prod_{j=1}^{n}(1-\eta_{Uj}^+)^{\omega_j}\right]\ge\left\{1-\prod_{j=1}^{n}(1-\eta_{Lj}^{+\,*})^{\omega_j},1-\prod_{j=1}^{n}(1-\eta_{Uj}^{+\,*})^{\omega_j}\right\}$$

$$\left[\bigcup_{\eta_{Lj}^+\in f_{LAj}^+}\left\{1-\prod_{j=1}^{n}(1-\eta_{Lj}^+)^{\omega_j}\right\},\bigcup_{\eta_{Lj}^+\in f_{LAj}^+}\left\{1-\prod_{j=1}^{n}(1-\eta_{Uj}^+)^{\omega_j}\right\}\right]$$

$$\ge\left[\bigcup_{\eta_{Lj}^+\in f_{LAj}^+}\left\{1-\prod_{j=1}^{n}(1-\eta_{Lj}^{+\,*})^{\omega_j}\right\},\bigcup_{\eta_{Lj}^+\in f_{LAj}^+}\left\{1-\prod_{j=1}^{n}(1-\eta_{Uj}^{+\,*})^{\omega_j}\right\}\right]$$

Since $t_{LAj}^-\ge t_{LAj}^{-\,*}$for all j, then we have:

$$\gamma_{Lj}^-\ge\gamma_{Lj}^{-\,*},\gamma_{Uj}^-\ge\gamma_{Uj}^{-\,*},-\gamma_{Lj}^-\le-\gamma_{Lj}^{-\,*},-\gamma_{Uj}^-\le-\gamma_{Uj}^{-\,*}$$
$$1-(-\gamma_{Lj}^-)\ge1-(-\gamma_{Lj}^{-\,*}),1-(-\gamma_{Uj}^-)\ge1-(-\gamma_{Uj}^{-\,*})$$

$$\left[\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^-)\right)^{\omega_j},\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^-)\right)^{\omega_j}\right]\ge\left[\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-\,*})\right)^{\omega_j},\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-\,*})\right)^{\omega_j}\right]$$

$$\left[1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-})\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-})\right)^{\omega_j}\right]$$
$$\leq\left\{1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-*})\right)^{\omega_j},1-\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-*})\right)^{\omega_j}\right\}$$
$$\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-})\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-})\right)^{\omega_j}\right)\right\}$$
$$\geq\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-*})\right)^{\omega_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-*})\right)^{\omega_j}\right)\right\}$$
$$\left\{\bigcup_{\gamma_{Lj}^{-}\in h_{T_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-})\right)^{\omega_j}\right)\right\},\bigcup_{\gamma_{Uj}^{-}\in h_{T_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Uj}^{-})\right)^{\omega_j}\right)\right\}\right\}$$
$$\geq\left\{\bigcup_{\gamma_{Lj}^{-}\in h_{T_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-*})\right)^{\omega_j}\right)\right\},\bigcup_{\gamma_{Uj}^{-}\in h_{T_j}^{-}}\left\{-\left(1-\prod_{j=1}^{n}\left(1-(-\gamma_{Lj}^{-*})\right)^{\omega_j}\right)\right\}\right\}$$

Since $i_{LAj}^{-}\geq i_{LAj}^{-}{}^{*}$ and $i_{UAj}^{-}\geq i_{UAj}^{-}{}^{*}$ for all j, then we have $\delta_{Lj}^{-}\geq\delta_{Lj}^{-*}$, $\delta_{Uj}^{-}\geq\delta_{Uj}^{-*}$, $-\delta_{Lj}^{-}\leq-\delta_{Lj}^{-*}$, $-\delta_{Uj}^{-}\leq-\delta_{Uj}^{-*}$.

$$\left\{\prod_{j=1}^{n}(-\delta_{Lj}^{-})^{\omega_j},\prod_{j=1}^{n}(-\delta_{Uj}^{-})^{\omega_j}\right\}\leq\left\{\prod_{j=1}^{n}(-\delta_{Lj}^{-*})^{\omega_j},\prod_{j=1}^{n}(-\delta_{Uj}^{-*})^{\omega_j}\right\}$$
$$\left\{-\prod_{j=1}^{n}(-\delta_{Lj}^{-})^{\omega_j},-\prod_{j=1}^{n}(-\delta_{Uj}^{-})^{\omega_j}\right\}\geq\left\{-\prod_{j=1}^{n}(-\delta_{Lj}^{-*})^{\omega_j},-\prod_{j=1}^{n}(-\delta_{Uj}^{-*})^{\omega_j}\right\}$$
$$\left\{\bigcup_{\delta_{Lj}^{-}\in i_{LAj}^{-}}\left\{-\prod_{j=1}^{n}(-\delta_{Lj}^{-})^{\omega_j}\right\},\bigcup_{\delta_{Uj}^{-}\in i_{UAj}^{-}}\left\{-\prod_{j=1}^{n}(-\delta_{Uj}^{-})^{\omega_j}\right\}\right\}$$
$$\geq\left\{\bigcup_{\delta_{Lj}^{-}\in i_{LAj}^{-}}\left\{-\prod_{j=1}^{n}(-\delta_{Lj}^{-*})^{\omega_j}\right\},\bigcup_{\delta_{Uj}^{-}\in i_{LAj}^{-}}\left\{-\prod_{j=1}^{n}(-\delta_{Uj}^{-*})^{\omega_j}\right\}\right\}$$

Since $f_{LAj}^{-}\leq f_{LAj}^{-}{}^{*}$ for all j, then we have $\eta_{Lj}^{-}\leq\eta_{Lj}^{-*}$, $\eta_{Uj}^{-}\leq\eta_{Uj}^{-*}$, $-\eta_{Lj}^{-}\geq-\eta_{Lj}^{-*}$, $-\eta_{Uj}^{-}\geq-\eta_{Uj}^{-*}$

$$\left\{\prod_{j=1}^{n}(-\eta_{Lj}^{-})^{\omega_j}\right\}\geq\left\{\prod_{j=1}^{n}(-\eta_{Lj}^{-*})^{\omega_j}\right\}$$
$$\left\{-\prod_{j=1}^{n}(-\eta_{Lj}^{-})^{\omega_j}\right\}\leq\left\{-\prod_{j=1}^{n}(-\eta_{Lj}^{-*})^{\omega_j}\right\}$$
$$\left\{\bigcup_{\gamma_{F_j}^{-}\in h_{F_j}^{-}}\left\{-\prod_{j=1}^{n}(-\eta_{Lj}^{-})^{\omega_j}\right\}\right\}\leq\left\{\bigcup_{\gamma_{F_j}^{-}\in h_{F_j}^{-}}\left\{-\prod_{j=1}^{n}(-\eta_{Lj}^{-*})^{\omega_j}\right\}\right\}$$

Based on the above analysis, we have:

$$IVBNHFWG_{\omega}(A_1,A_2,\ldots,A_n)\leq IVBNHFWG_{\omega}^{*}(A_1^{*},A_2^{*},\ldots,A_n^{*})$$

which completes the proof of the ii).

iii) Boundedness

Similar to the proof of ii), thus omitted. □

Definition 3.5. (Score, accuracy and certainty functions)

Let A_1 be an IVBNHFE in X $=\{x_1,x_2,\ldots,x_n\}$;

$A_1 = \{\langle x_i, t_1^+(x_i), i_1^+(x_i), f_1^+(x_i), t_1^-(x_i), i_1^-(x_i), f_1^-(x_i)\rangle | \, x_i \in X, i = 1,2,\dots,n\}$; the score function, accuracy function and certainty function are effective tools to rank IVBNHFEs, and here we give definitions of these functions. Then, the score functions $s(A_1)$ and the accuracy function $a(A_1)$ and certainty function $c(A_1)$ for IVBNHFS are proposed as follows:

$$s(A_1) = \frac{1}{12}\begin{bmatrix} \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{L1i}^{+} + \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{U1i}^{+} + 1 - \frac{1}{p_i}\sum_{k=1}^{p_i}\delta_{L1i}^{+} \\ +1 - \frac{1}{p_i}\sum_{k=1}^{p_i}\delta_{U1i}^{+} + 1 - \frac{1}{q_i}\sum_{k=1}^{q_i}\eta_{L1i}^{+} + 1 - \frac{1}{q_i}\sum_{k=1}^{q_i}\eta_{U1i}^{+} \\ +1 + \frac{1}{r_i}\sum_{k=1}^{r_i}\gamma_{L1i}^{-} + 1 + \frac{1}{r_i}\sum_{k=1}^{r_i}\gamma_{U1i}^{-} \\ -\frac{1}{s_i}\sum_{k=1}^{s_i}\delta_{L1i}^{-} - \frac{1}{s_i}\sum_{k=1}^{s_i}\delta_{U1i}^{-} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{L1i}^{-} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{U1i}^{-} \end{bmatrix} \tag{11}$$

$$a(A_1) = \begin{matrix} \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{L1i}^{+} + \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{U1i}^{+} - \frac{1}{q_i}\sum_{k=1}^{q_i}\eta_{L1i}^{+} - \frac{1}{q_i}\sum_{k=1}^{q_i}\eta_{U1i}^{+} \\ +\frac{1}{r_i}\sum_{k=1}^{r_i}\gamma_{L1i}^{-} + \frac{1}{r_i}\sum_{k=1}^{r_i}\gamma_{U1i}^{-} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{L1i}^{-} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{U1i}^{-} \end{matrix} \tag{12}$$

$$c(A_1) = \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{L1i}^{+} + \frac{1}{l_i}\sum_{k=1}^{l_i}\gamma_{U1i}^{+} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{L1i}^{-} - \frac{1}{t_i}\sum_{k=1}^{t_i}\eta_{U1i}^{-} \tag{13}$$

where:
$t_1^+(x) = \{\gamma_1^+ | \gamma_1^+ \in t_1^+(x)\}, i_1^+(x) = \{\delta_1^+ | \delta_1^+ \in i_1^+(x)\}$, $f_1^+(x) = \{\eta_1^+ | \eta_1^+ \in i_1^+(x)\}$, $t_1^-(x) = \{\gamma_1^- | \gamma_1^- \in t_1^-(x)\}, i_1^-(x) = \{\delta_1^- | \delta_1^- \in i_1^-(x)\}$, $f_1^-(x) = \{\eta_1^- | \eta_1^- \in i_1^-(x)\}$, $\gamma_{1i}^+ = [\gamma_{L1i}^+, \gamma_{U1i}^+] \in t_1^+(x_i)$, $\delta_{1i}^+ = [\delta_{L1i}^+, \delta_{U1i}^+] \in i_1^+(x_i), \eta_{1i}^+ = [\eta_{L1i}^+, \eta_{U1i}^+] \in f_1^+(x_i), \gamma_{1i}^- = [\gamma_{L1i}^-, \gamma_{U1i}^-] \in t_1^-(x_i), \delta_{1i}^- = [\delta_{L1i}^-, \delta_{U1i}^-] \in i_1^-(x_i)$, $\eta_{1i}^- = [\eta_{L1i}^-, \eta_{U1i}^-] \in f_1^-(x_i)$ for $(i = 1,2,\dots,n)$, and, $l_i, p_i, q_i, r_i, s_i, t_i$, are the number of intervals in $t_1^+(x_i), i_1^+(x_i)$, $f_1^+(x_i), t_1^-(x_i), i_1^-(x_i)$, and $f_1^-(x_i)$, respectively.

Based on the functions $s(A)$ and $a(A)$, two IVBNHFEs can be compared and ranked. If score values $s(A)$ are different, we can rank them; if some of their score values are same, we continue to calculate accuracy values $a(A)$. Finally, if some of their accuracy values present the same value $a(A)$, we continue to calculate certainty values $c(A)$.

We define the comparative relations for $s(A)$, $a(A)$ and $c(A)$ as follows:

i) If $s(A_1) > s(A_2)$, then $A_1$is greater than A_2, denoted by $A_{2HZS1} > A_{2HZS2}$;
ii) If $s(A_1) = s(A_2)$ and $a(A_1) > a(A_2)$, then A_1 is bigger than A_2, represented by $A_1 > A_2$;
iii) If $s(A_1) = s(A_2)$ and $a(A_1) = a(A_2)$, and $c(A_1) > c(A_2)$, then A_1 is greater than A_2, denoted by $A_1 > A_2$.

IV. Proposed IVBNHFS-MADM Approach based on the IVBNHFWA or IVBNHFWG Operators

In this section, we provide an IVBNHFS-MADM approach based on the IVBNHFWA or IVBNHFWG operators to deal with IVBNHF information. In a MADM problem, assume that $A_i = A_1, A_2, \dots, A_m$ is a set of m alternatives and $C_j = C_1, C_2, \dots, C_n$ is a set of n attributes. Suppose that the weight vector of C is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$. The evaluation value of an alternative under an attribute can be expressed using an IVBNHFE. Then, we can construct a decision matrix with IVBNHF information, and provide the following IVBNHFS-MADM procedures on the proposed IVBNHFWA or IVBNHFWG operators.

Algorithm 4.1.
Step 1. Build the choice grid (decision matrix) as:

$$(p_{ij})_{m\times n} = \left(A_i(C_j)\right)_{m\times n} = \left\{\langle C_j, t_i^+(C_j), i_i^+(C_j), f_i^+(C_j), t_i^-(C_j), i_i^-(C_j), f_i^-(C_j)\rangle \mid C_j \in C, j = 1,2,\dots,n\right\}_{mxn}$$

where $t_i^+(C_j) = \left\{\gamma^+{}_{ij} | \, \gamma^+{}_{ij} \in t_i^+(C_j)\right\}$, $i_i^+(C_j) = \{\delta^+{}_{ij} | \, \delta^+{}_{ij} \in i_i^+(C_j)\}$, and $f_i^+(C_j) = \left\{\eta^+{}_{ij} | \, \eta^+{}_{ij} \in f_i^+(C_j)\right\}$, $t_i^-(C_j) = \left\{\gamma^-{}_{ij} | \, \gamma^-{}_{ij} \in t_i^-(C_j)\right\}$, $i_i^-(C_j) = \{\delta^-{}_{ij} | \, \delta^-{}_{ij} \in i_i^-(C_j)\}$, and $i_i^-(C_j) = \left\{\eta^-{}_{ij} | \, \eta^-{}_{ij} \in i_i^-(C_j)\right\}$. And $\gamma^+{}_{ij} = [\gamma_{Lij}^+, \gamma_{Uij}^+] \in [0,1], \delta^+{}_{ij} = [\delta_{Lij}^+, \delta_{Uij}^+] \in [0,1], \eta^+{}_{ij} = [\eta_{Lij}^+, \eta_{Uij}^+] \in [0,1], \gamma^-{}_{ij} = [\gamma_{Lij}^-, \gamma_{Uij}^-] \in [-1,0], \delta^-{}_{ij} = [\delta_{Lij}^-, \delta_{Uij}^-] \in, [-1,0]$ and $\eta^-{}_{ij} = [\eta_{Lij}^-, \eta_{Uij}^-] \in [-1,0]$.

Step 2. Calculate the aggregated value for each alternative using the IVBNHFWA or IVBNHFWG operators as: $p_{ia} = \text{IVBNHFWA}(p_{i1}, p_{i2}, \dots, p_{in})$ or $p_{ig} = \text{IVBNHFWG}(p_{i1}, p_{i2}, \dots, p_{in})$ for each $p_i(i = 1,2,\dots,m)$

Step 3. Obtain the score values $s(p_{ia})$ or $s(p_{ig})$, accuracy function $a(p_{ia})$ or $a(p_{ig})$ and certainty function $c(p_{ia})$ or $c(p_{ig})$, if necessary, of the collective IVBNHFE by Equations (11), (12) and (13).

Step 4. Rank all the alternatives corresponding to the values of the results in the step 3, and select the best one(s) based on the largest value of score values $s(p_{ia})$ or $s(p_{ig})$.
Step 5. End

V. ILLUSTRATIVE EXAMPLE

This section introduces an illustrative case adapted from reference [38] to demonstrate the application of the above IVBNHFS-MADM strategy. A company wants to invest some money in one of the four possible alternatives $\boldsymbol{A_i}(\boldsymbol{i} = \mathbf{1, 2, 3, 4})$. $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, and $\boldsymbol{A_4}$, represent a car company, a food company, a computer company, and an arms company, respectively.

The four alternatives need to be evaluated according to the four attributes $\boldsymbol{C_j}$ $(\boldsymbol{j} = \mathbf{1, 2, 3, 4})$. $\boldsymbol{C_1}$, $\boldsymbol{C_2}$, $\boldsymbol{C_3}$ and $\boldsymbol{C_4}$ represent respectively the risk, the growth, the environmental impact, and the performance. Corresponding to the four attributes, the weight vector is $\boldsymbol{\omega} = (\mathbf{0.24, 0.26, 0.26, 0.24})^{\boldsymbol{T}}$.

Step 1. An IVBNHF decision matrix $(p_{ij})_{4\times4}$ is constructed, shown in Table I.

TABLE I. IVBNHF DECISION MATRIX.

	$\boldsymbol{C_1}$
$\boldsymbol{A_1}$	{{[0.5, 0.6]}, {[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]}, {[0.1, 0.2], [0.2, 0.3], [0.3, 0.4], [0.4, 0.5], [0.5, 0.6], [0.6, 0.7]}, {[-0.2, -0.1]}, {[-0.6, -0.5], [-0.5, -0.4], [-0.4, -0.3], [-0.3, -0.2]}, {[-0.4, -0.3]}}
$\boldsymbol{A_2}$	{{[0.1, 0.2]}, {[0.3, 0.4], [0.4, 0.5], [0.5, 0.6], [0.6, 0.7], [0.7, 0.8]}, {[0.2, 0.3], [0.3, 0.4]}, {[−0.5, −0.4], [−0.4, −0.3],[−0.3, −0.2]}, {[−0.9, −0.8], [−0.8, −0.7], [−0.7, −0.6], [−0.6, −0.5], [−0.5, −0.4], [−0.4, −0.3]}, {[−0.6, −0.5], [−0.5, −0.4], [−0.4, −0.3], [−0.3, −0.2], [−0.2, −0.1]}}
$\boldsymbol{A_3}$	{{[0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]}, {[0.4,0.5], [0.5,0.6]}, {[0.4,0.5], [0.5,0.6]}, {[-0.3,-0.2]}, {[-0.7,-0.6],[-0.6,-0.5]}, {[-0.5,-0.4]}}
$\boldsymbol{A_4}$	{{[0.6,0.7], [0.7,0.8], [0.8,0.9]}, {[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]}, {[0.5,0.6]}, {[-0.8,-0.7],[-0.7,-0.6], [-0.6,-0.5]}, {[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2], [-0.2,-0.1]}, {[-0.2,-0.1]}}
	$\boldsymbol{C_2}$
$\boldsymbol{A_1}$	{{[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8],[0.8,0.9]}, {[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]},{[0.2,0.3],[0.3,0.4], [0.4,0.5]}, {[-0.8,-0.7]}, {[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2], [-0.2,-0.1]}, {[-0.4,-0.3], [-0.3,-0.2], [-0.2,-0.1]}}
$\boldsymbol{A_2}$	{{[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]}, {[0.1,0.2],[0.2,0.3],[0.3,0.4]}, {[0.3,0.4]},{[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2], [-0.2,-0.1]}, {[-0.3,-0.2], [-0.2,-0.1]}, {[-0.9,-0.8], [-0.8,-0.7],[-0.7,-0.6], [-0.6,-0.5], [-0.5,-0.4]}}
$\boldsymbol{A_3}$	{{[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6]}, {[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8],[0.8,0.9]}, {[0.3,0.4], [0.4,0.5]}, {[-0.8,-0.7]}, {[-0.4,-0.3]}, {[-0.7,-0.6]}}
$\boldsymbol{A_4}$	{{[0.1,0.2]}, {[0.8,0.9]},{[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7]}, {[-0.5,-0.4]}, {[-0.6,-0.5], [-0.5,-0.4], [-0.4,-0.3]}, {[-0.5,-0.4], [-0.4,-0.3]}}
	$\boldsymbol{C_3}$
$\boldsymbol{A_1}$	{{[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6]},{[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5]}, {[0.1,0.2],[0.2,0.3],[0.3,0.4]}, {[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2]}, {[-0.7,-0.6], [-0.6,-0.5], [-0.5,-0.4], [-0.4,-0.3]}, {[-0.4,-0.3], [-0.3,-0.2]}}
$\boldsymbol{A_2}$	{{[0.3,0.4]}, {[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6]}, {[0.5,0.6], [0.6,0.7]}, {[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2], [-0.2,-0.1]}, {[-0.8,-0.7]}, {[-0.9,-0.8]}}
$\boldsymbol{A_3}$	{{[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8],[0.8,0.9]}, {[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]}, {[0.2,0.3]}, {[-0.5,-0.4]}, {[-0.6,-0.5]}, {[-0.7,-0.6]}}
$\boldsymbol{A_4}$	{{[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7]}, {[0.5,0.6], [0.6,0.7], [0.7,0.8]}, [0.8,0.9]}, {[-0.9,-0.8]}, {[-0.8,-0.7],[-0.7,-0.6],[-0.6,-0.5]},{[-0.5,-0.4], [-0.4,-0.3], [-0.3,-0.2]}}
	$\boldsymbol{C_4}$
$\boldsymbol{A_1}$	{{[0.6,0.7], [0.7,0.8]}, {[0.4,0.5], [0.5,0.6]},{[0.1,0.2],[0.2,0.3]}, {[-0.4,-0.3]}, {[-0.6,-0.5], [-0.5,-0.4], [-0.4,-0.3]}, {[-0.7,-0.6],[-0.6,-0.5]}}
$\boldsymbol{A_2}$	{{[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8]},{[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7], [0.7,0.8],[0.8,0.9]},{[0.1,0.2]}, {[-0.8,-0.7],[-0.7,-0.6]}, {[-0.6,-0.5], [-0.5,-0.4]}, {[-0.4,-0.3], [-0.3,-0.2]}}
$\boldsymbol{A_3}$	{{[0.7,0.8],[0.8,0.9]}, {[0.1,0.2],[0.2,0.3],[0.3,0.4]},{[0.2,0.3], [0.3,0.4], [0.4,0.5], [0.5,0.6]}, {[-0.7,-0.6]}, {[-0.9,-0.8],[-0.8,-0.7],[-0.7,-0.6],[-0.6,-0.5]}, {[-0.3,-0.2]}}
$\boldsymbol{A_4}$	{{[0.4,0.5], [0.5,0.6]}, [0.3,0.4], [0.4,0.5]},{[0.1,0.2],[0.2,0.3],[0.3,0.4], [0.4,0.5], [0.5,0.6], [0.6,0.7]}, {[-0.3,-0.2], [-0.2,-0.1]}, {[-0.6,-0.5]}, {[-0.7,-0.6],[-0.6,-0.5],[-0.5,-0.4], [-0.4,-0.3]}}

Step 2. Calculate the aggregated value for each alternative using the IVBNHFWA or IVBNHFWG operators as: $p_{ia} =$ IVBNHFWA$(p_{11}, p_{12}, p_{13}, p_{14})$ or $p_{ig} =$ IVBNHFWG$(p_{11}, p_{12}, p_{13}, p_{14})$ for each $p_i(i = 1,2,3,4)$

For the IVBNHFWA, we have:
$p_{1a} = \{\{[0.397335317, 0.503263331], [0.437541551, 0.549324117], [0.415511345, 0.520213179] \ldots.$
$[-0.413357639, -0.310705404], [-0.430105733, -0.323567639], [-0.389367964, -0.286354092]\}\}$
$p_{2a} = \{\{[0.230212918, 0.33082037], [0.260455258, 0.36180178], [0.294694404, 0.397774716] \ldots..$
$[-0.623208823, -0.492121534], [-0.615204399, -0.484262328], [-0.600701823, -0.467466512]\}\}$
$p_{3a} = \{\{[0.412383551, 0.524530436], [0.466871325, 0.597398376], [0.435469021, 0.546543535] \ldots.$
$[-0.585181209, -0.48274477], [-0.555528705, -0.454288264], [-0.584396321, -0.479322162]\}\}$
$p_{4a} = \{\{[0.348135775, 0.454504981], [0.376044456, 0.482950186], [0.370378948, 0.475935642] \ldots..$
$[-0.392871234, -0.289932083], [-0.359470386, -0.258171713], [-0.33082037, -0.230212918]\}\}$
For the IVBNHFWG, we have:
$p_{1g} = \{\{[0.300995933, 0.421098046], [0.31234018, 0.434811772], [0.36043622, 0.467914985] \ldots.$
$[-0.36817483, -0.254866895], [-0.354517153, -0.239625593], [-0.34164103, -0.229366349]\}\}$
$p_{2g} = \{\{[0.207409396, 0.314288667], [0.223517997, 0.333062183], [0.236869477, 0.349230727] \ldots..$
$[-0.43368927, -0.308164634], [-0.44317663, -0.320514805], [-0.413610498, -0.290794503]\}\}$
$p_{3g} = \{\{[0.412383551, 0.416195897], [0.305694427, 0.428128757], [0.319046162, 0.441056674] \ldots$
$\{[-0.56031823, -0.457396491], [-0.539967378, -0.437813682]\}, \{[-0.526881419, -0.418196359]\}\}$
$p_{4g} = \{\{[0.256754756, 0.374020715], [0.270879975, 0.390750151], [0.285300297, 0.403069306] \ldots..$
$[-0.346410162, -0.234461817], [-0.331579081, -0.222235648], [-0.314288667, -0.207409396]\}\}$

Step 3. Based on the score function of IVBNHFEs in Eq. (11), we get:
$s(p_{1a}) = 0.5730$, $s(p_{2a}) = 0.5905$, $s(p_{3a}) = 0.5828$, and $s(p_{4a}) = 0.4713$.
$s(p_{1g}) = 0.5172$, $s(p_{2g})$ =0.4969, $s(p_{3g}) = 0.5206$, and $s(p_{4g}) = 0.3787$.

Step 4. Ranking the alternatives A_i according to the score values.
From the results of step 3, the ranking order of the four alternatives is $A_2 > A_3 > A_1 > A_4$ for the IVBNHFWA and $A_3 > A_1 > A_2 > A_4$ for the IVBNHFWG model. Therefore, the best alternative is A_2 or A_3.

Step 5. End

VI. Comparison with Existing Methods

To further validate the feasibility of above MADM methods, a comparison analysis was conducted with other method. To be specific, the comparative study was based on the same illustrative example [38] in which the weight of attributes is ω = (0.24, 0.26, 0.26, 0.24). Then, the results by utilizing different approaches with complete weight information are shown in Table II.

TABLE II. RESULTS OBTAINED BY UTILIZING THE DIFFERENT METHODS

Methods		Final Ranking	Best Alternative	Worst Alternative
Interval Valued Bipolar Neutrosophic Sets [38]		$A_2 > A_3 > A_1 > A_4$	A_2	A_4
Our Method	IVBNHFWA	$A_2 > A_3 > A_1 > A_4$	A_2	A_4
	IVBNHFWG	$A_3 > A_1 > A_2 > A_4$	A_3	A_4

We can see the comparison analysis with the Fig. 1.

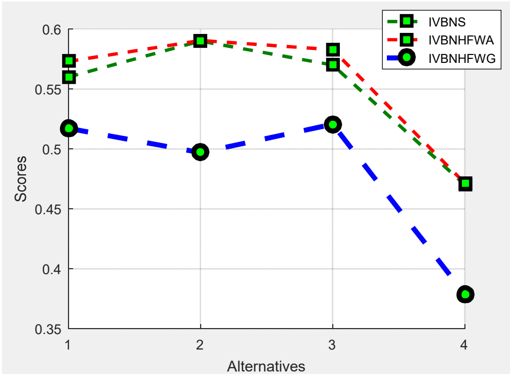

Fig. 1. Comparison with existing methods

For different methods: Deli 's methods in [38], IVBNHFWA or IVBNHFWG, as we can see from Table II and Fig.1, the final ranking may be different each other. For the compared method in [38], Deli and al. propose two kinds of aggregation operators, the Interval Valued Bipolar Neutrosophic Weighted Average (IVBNWA) and Interval Valued Bipolar Neutrosophic Weighted Geometric (IVBNWG) operators, which are also applied to MADM problems with Interval Valued Bipolar Neutrosophic information.

Thus, according to the results presented in Table II and Fig.1, if the Interval Valued Bipolar Neutrosophic Sets operators [38] are used, the desirable alternative is A_2, and if our method is utilized, the best choice is A_2 or A_3. Then, the IVBNHFWA ranking order of the four alternatives is in agreement with the results of Deli 's method [38]. On the other hand, for all the compared methods, the worst alternative is always A_4. Therefore, for the same IVBNHF information, the results obtained by the proposed methods (IVBNHFWA or IVBNHFWG) in this paper are consistent and accurate with those obtained using the IVBNS compared method in [38], which further demonstrates the practicability in application of the IVBNHFWA and IVBNHFWG operators.

VII. Conclusion

In this paper, we have developed some new ideas of Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Average (IVBNHFWA) and Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Weighted Geometric (IVBNHFWG) operators to aggregate the IVBNHFS. Also, we defined the score functions $s(A)$, the accuracy function $a(A)$ and the certainty function $c(A)$. Moreover, this paper presented the developed aggregation operators to IVBNHFS-MADM with IVBNHF information. Finally, an illustrative numerical example is used to confirm the validity of the IVBNHFWA and IVBNHFWG operators approaches. The advantages of this new approaches are that: (1) it is reliable and reasonable to aggregate the IVBNHFS under the IVBNHF information; (2) it offers an effective and powerful mathematics tool for handling the IVBNHFS-MADM under uncertainty and can provide more reliable and flexible aggregation results in IVBNHFS-MADM; (3) it not only considers relationships between the alternatives or the attributes, but also takes into account the connection between alternative and itself or the interrelations between the attribute and itself, furthermore, interrelationships between alternative or criteria are handled once. The new operators' strategies provide some sensible and solid IVBNHFS-MADM aggregation operators, which expand the determination extent of the leader. Thus, in future work, we will study more aggregation operators for IVBNHFS such as Hamacher, Einstein, Choquet, Dombi, .We will apply these to IVBNHFS- MADM, data mining, game theory, medical diagnosis, fault diagnosis and pattern recognition.

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