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First Aboodh Transform of Fractional Order and Its Properties

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Abstract— **In this paper, we propose definitions of a fractional first Aboodh transform of order α, where α ϵ [0, 1], for a functions which are fractional differentiable. We then present some main properties of this transform. Furthermore, we prove some related theorems.**

***Keywords*— Aboodh transform , Laplace transform , double Aboodh transform , Mittag leffler function .**

I. Introduction

Laplace , Sumudu, Natural, Ezaki, Mellin, Aboodh, and so on, are examples of different kinds of integral transforms which are used in many areas of physics, statistics, astronomy engineering and various domains of applied sciences[1,2,3,4].
One of the most familiar method for solving both ordinary and fractional differential equations is the integral transform method, due to that many researchers are doing great efforts to study and generalize these transforms [5,6,7].

The Aboodh transform [8, 9] is a new transform derived from the Fourier transform and similar Laplace transform. Aboodh transform is defined for a function of exponential order in the time domain t≥0. Aboodh transform and some of its properties are used to solve partial differential equations of integer fractional order [10, 11].
Our objective in this article is to define fractional Aboodh transform and prove some related properties and theorems.
This study has been organized as follows: In Section II we recall some definitions and results related to Aboodh transform, in section III we introduce the definition of fractional Aboodh transform**.** In section IV, we present and prove some properties of fractional Aboodh transform, in section V, we state the convolution theorem of the fractional Aboodh transform and its proof. Some theorems and properties of the fractional Aboodh transform are given in section VI and VII. In section VIII, we give the inversion theorem and its proof. Finally, the conclusion follows in section 7.

II. Preliminaries

In this section , we present definition of Aboodh transform and some known resulats .
Definition 1: Aboodh transform of continuos afunction $f(x)$ is defined by

$$A\left[f(x)\right]=k(p)=\frac{1}{p}\int_{0}^{\infty}e^{-px}f(x)dx \ ,where\ x>0 \qquad (1)$$

And

$$f(x)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}pe^{px}k(p)dp \qquad (2)$$

is the inverse of Aboodh transform .
Defintion 2: (Modifide Rieman –Liouvile derivative) let $g(x)$ be a continuous function operator $FW(h)$ as following:

$$\Delta^{\alpha}g(x)=(FW-h)^{\alpha}g(x)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j}g(\gamma+(\alpha-k)h)$$

Where h is apositive real number and denote aconstant discretization span .
Then the fractional difference of order $\alpha,\alpha<1$ of $g(x)$ is defined by

$$\Delta^{\alpha} g(x) = (FW - h)^{\alpha} = \sum_{j=0}^{\infty} (-1)^{j} \left(j^{\alpha}\right) g\left(\gamma + (\alpha - k)h\right)$$

And its α-derivative is defined by :

$$g^{\alpha}(x) = \lim_{h \to 0} \frac{\Delta^{\alpha} g(x)}{h^{\alpha}}$$

For more details see [12].

Definition 3: Let $g(x)$ be the function that defined in definition 2, then

(1) If $g(x) = c$, where c is constant , then its α-derivative of order α is given by :

$$D_x^{\alpha} g(x) = \begin{cases} \frac{c}{\Gamma(1-\alpha) x^{\alpha}}, & \alpha \le 0 \\ 0, & 0 \end{cases}$$

(2) If g(x) is not constant , hence

$$g(x) = g(0) + \left(g(x) - g(0)\right)$$

with fractional derivative given by:

$$g^{\alpha}(x) = D_x^{\alpha} g(0) + D_x^{\alpha}\left(g(x) - g(0)\right)$$

for α >0, we will put

$$D_x^{\alpha}\left(g(x) - g(0)\right) = D_x^{\alpha} g(x) = D_x\left(g^{\alpha-1}(x)\right)$$

If, $k < \alpha < k+1$, we will put

$$g^{\alpha}(x) = \left(g^{\alpha-k}(x)\right)^{k}, k \le \alpha \le k+1, k \ge 1$$

Lemma 1: The solution of fractional differential equation $dy = g(x)(dx)^{\alpha}, x > 0, y(0) = 0,$ is defined by:

$$y(x) = \int_0^x g(u)(du)^{\alpha} \quad , y(0) = 0$$

$$= \alpha \int_0^x (x-u)^{\alpha-1} g(u) du \qquad 0 < \alpha < 1$$

Where the integration with respect to $(dx)^{\alpha}$.

for more result see [13] .

Defintion 4: The Mittag-Leffler function is defined by:

$$E_{\alpha}(x) = \sum_{j=0}^{\infty} \frac{x^{j}}{\Gamma(\alpha j + 1)}, \ x \in \mathbb{C}, \ \Re(\alpha) > 0.$$

Defintion 5: The fractional laplace transform of function g(x) is given by[14] :

$$L_{\alpha}\{g(x)\} = \int_0^{\infty} E_{\alpha}\left(-p^{\alpha} x^{\alpha}\right) g(x)(dx)^{\alpha}, p \in \mathbb{C}, x > 0. \quad (3)$$

III. Aboodh Transform Of Fractional Order

Definition 6: Let g(x) be a function where x>0, then the Aboodh transform of fractional order α is defined by :

$$A_{\alpha}\left[g(x)\right] = K_{\alpha}(P) = \frac{1}{p}\int_0^{\infty} E_{\alpha}(-(px)^{\alpha}) g(x)(dx)^{\alpha} \quad (4)$$

$$= \lim_{V \to \infty} \frac{1}{P}\int_0^{V} E_{\infty}(-(px)^{\alpha}) g(x)(dx)^{\alpha} ,$$

where $p \in \mathbb{C}$, x>0.

Remark 1: when $\alpha = 1$, Fractional Aboodh transfrom (4) turns to Aboodh transform (1).

IV. Some Properties Of Fractional First Aboodh Transform .

1) Linearity property.

Let $f(x)$, $g(x)$ be two functions, then:

$$A_{\alpha}\{af(x) + b\,g(x)\} = aA_{\alpha}\{f(x)\} + bA_{\alpha}\{g(x)\},$$

where a, b are constants.

Proof:

By applying the definition of factional Aboodh transform, we can simply get the proof.

2) Changing of scale.

If $A_{\alpha}\{g(x)\} = k_{\alpha}(p)$, then: $A_{\alpha}\{g(ax)\} = \frac{1}{a^{\alpha}} k_{\alpha}\left(\frac{p}{a}\right)$,

Where a, is constant.

Proof: $A_{\alpha}\{g(ax)\} = \frac{1}{p}\int_0^{\infty} E_{\alpha}(-(px)^{\alpha}) g(ax)(dx)^{\alpha}$.

By letting $u = ax$, we get:

$$A_{\alpha}\{g(ax)\} = \frac{1}{a^{\alpha}}\frac{1}{p}\int_0^{\infty} E_{\alpha} - \left(\frac{p}{a}u\right)^{\alpha}) g(u)(du)^{\alpha} = \frac{1}{a^{\alpha}} k_{\alpha}\left(\frac{p}{a}\right).$$

3) Shifting property.

If $A_{\alpha}\{g(x)\} = k_{\alpha}(p)$, then

$A_{\alpha}\left\{E_{\alpha}(-ax)^{\alpha}\right\} g(x)\} = k_{\alpha}(p+a)$, where a, is constant.

Proof :

$$A_{\alpha}\left\{E_{\alpha}(-ax)^{\alpha} g(x)\right\} = \frac{1}{p}\int_0^{\infty} E_{\alpha}\left(-(px)^{\alpha}\right) E_{\alpha}\left(-(ax)^{\alpha}\right) g(x)(dx)^{\alpha},$$

by using the following property of the Mittag-Leffler function,

$$E_{\alpha}\left(-(px)^{\alpha}\right) E_{\alpha}\left(-(ax)^{\alpha}\right) = E_{\alpha}\left(-(p+a)x)^{\alpha}\right)$$

we get:

$$A_\alpha\left\{E_\alpha(-ax)^\alpha g(x)\right\}=\frac{1}{p}\int_0^\infty E_\alpha\left(-(p+a)x\right)^\alpha g(x)(dx)^\alpha$$
$$=k_\alpha(p+a)$$

4) Multiplication by x^α .

If $A_\alpha\{g(x)\}=k_\alpha(p)$, then

$$A_\alpha\left(x^\alpha g(x)\right)=\frac{1}{p}D_p^\alpha\left(pk_\alpha(p)\right).$$

Proof:

$$A_\alpha\left(x^\alpha g(x)\right)=\frac{1}{p}\int_0^\infty E_\alpha\left(-(px)^\alpha\right)x^\alpha g(x)(dx)^\alpha$$
$$=\frac{1}{p}\int_0^\infty D_p^\alpha E_\alpha\left(-(px)^\alpha\right)g(x)(dx)^\alpha$$
$$=\frac{1}{p}D_p^\alpha\left[\int_0^\infty E_\infty\left(-(px)^\alpha\right)(dx)^\alpha\right]$$
$$=\frac{1}{p}D_p^\alpha\left(p\,k_\alpha(p)\right).$$

5) *Fractional Aboodh transform of fractional derivative.*

$$A_\alpha\left(g^\alpha(x)\right)==p^\alpha K_\alpha(p)-\frac{\Gamma(1+\alpha)g(0)}{p}.$$

Proof:

By applying Fractional integration by part formula, we get:

$$=\frac{1}{p}[\Gamma(1+\alpha)g(x)E_\alpha(-(px))^\alpha\Big|_0^\infty$$
$$-\int_0^\infty D_x^\alpha E_\alpha\left(-(px)^\alpha\right)g(x)(dx)^\alpha]$$
$$=\frac{1}{p}[-\Gamma(1+\alpha)g(0)+p^\alpha\int_0^\infty E_\alpha\left(-(px)^\alpha\right)g(x)(dx)^\alpha]$$
$$=-\frac{\Gamma(1+\alpha)g(0)}{p}+p^\alpha A_\alpha\left(g(x)\right)$$
$$=p^\alpha K_\alpha(p)-\frac{\Gamma(1+\alpha)g(0)}{p}.$$

Remark 2 : All results above are sutable for Aboodh transform when $\alpha=1$.

V. Convolution Theorem Of Fractional Aboodh Transform

Propositional 1: The convolution of order α of functions $f(x)$ and $g(x)$ can be defined by the expression:

$$\left(f(x)*_\alpha g(x)\right)=\int_0^x f(x-z)g(z)(dz)^\alpha \qquad (5)$$

Therefore, the fractional Aboodh transform of (5) is given by:

$$A_\alpha\{f(x)*_\alpha g(x)\}=pA_\alpha\{f(x)\}A_\alpha\{g(x)\}$$

Proof:

$$A_\alpha\{f(x)*_\alpha g(x)\}=$$
$$\frac{1}{P}\int_0^\infty E_\alpha\left(-(px)^\alpha\right)\left[\int_0^x f(x-z)g(z)(dz)^\alpha\right](dx)^\alpha$$
$$=\frac{1}{P}\int_0^\infty\left[\begin{array}{c}E_\alpha\left(-(pz)^\alpha\right)E_\alpha\left(-(p(x-z)^\alpha\right)\\ \left[\int_0^x f(x-z)g(z)(dz)^\alpha\right](dx)^\alpha\end{array}\right], \qquad (7)$$

by letting $u=x-z$, (7) becomes:

$$=\frac{1}{P}\int_0^\infty\left[\begin{array}{c}E_\alpha\left(-(pz)^\alpha\right)E_\alpha\left(-(px)^\alpha\right)\\ \left(\int_0^\infty f(u)g(z)(dz)^\alpha\right)(du)^\alpha\end{array}\right]$$
$$=\left[\begin{array}{c}\left(\frac{1}{P}\int_0^\infty E_\alpha\left(-(pz)^\alpha\right)g(z)(dz)^\alpha\right)\\ \times\left(\int_0^\infty E_\alpha\left(-(pu)^\alpha\right)f(u)(du)^\alpha\right)\end{array}\right]$$
$$=pA_\alpha\{f(x)\}A_\alpha\{g(x)\}$$

VI. Inversion Formula For The Fractional Aboodh Transform

Definition 7: The Dirac's distripution $\delta_\alpha(x)$ of order α , where $\alpha\subset(0,1)$ is defined by:

$$\int_{\Re} f(x)\delta_\alpha(x-a)(dx)^\alpha=\alpha f(a). \qquad (8)$$

Lemma 2: Define the function

$$\delta_\alpha(x,\varepsilon)=\begin{cases}0 & ,x\notin[0,\varepsilon]\\ \varepsilon & ,x\in[0,\varepsilon]\end{cases}.$$

Then one has the limit

$$\lim_{\varepsilon\to0}\delta_\alpha(x,\varepsilon)=\delta_\alpha(x).$$

Proof: see [14]

Example 1 : The fractional Aboodh transform of $\delta_\alpha(x-a)$ can be calculated as follows:

$$A_\alpha\{\delta_\alpha(x-a)\}=\frac{1}{p}\int_0^\infty E_\alpha\left(-(px)^\alpha\right)\delta_\alpha(x-a)(dx)^\alpha$$
$$=\frac{\alpha}{p}E_\alpha\left(-(pa)^\alpha\right)$$

VII. Relation Between Dirac's Distripution $\delta_\alpha(x)$ And Mittag-Leffler Function $E_\alpha(x^\alpha)$

The following theorem clarify the relation between $\delta_\alpha(x)$ and $E_\alpha(x^\alpha)$, which can help us to prove inversion theorem that will consider later.

Theorem 1: [14]The following formula holds

$$\frac{\alpha}{(\mu_\alpha)^\alpha}\int_{\Re}E_\alpha\left(i(-ux)^\alpha\right)(du)^\alpha=\delta_\alpha(x). \qquad (9)$$

Where μ_α, satisfy $E_\alpha\left(i(\mu_\alpha)^\alpha\right)=1$, and called the period of the complex-valued Mittag-leffer function .

Proof: we test that (9) is consistent with

$$\int_{\Re}E_\alpha\left(i(vx)^\alpha\right)\delta_\alpha(x)(dx)^\alpha=\alpha. \qquad (10)$$

By substituting (9), in (10), we have:

$$\alpha=\int_{\Re}E_\alpha\left(i(vx)^\alpha\right)\left[\frac{\alpha}{(\mu_\alpha)^\alpha}\int_{\Re}E_\alpha\left(i(-ux)^\alpha\right)(du)^\alpha\right](dx)^\alpha$$
$$=\int_{\Re}(dx)^\alpha\frac{\alpha}{(\mu_\alpha)^\alpha}\int_{\Re}E_\alpha\left(i((v-u)x)^\alpha\right)(du)^\alpha$$
$$=\int_{\Re}(dx)^\alpha\frac{\alpha}{(\mu_\alpha)^\alpha}\int_{\Re}E_\alpha\left(i(-wx)^\alpha\right)(du)^\alpha$$
$$=\int_{\Re}\delta_\alpha(x)(dx)^\alpha$$
$$=\alpha$$

VIII. Inversion Theorem Of Aboodh Transform

In this section, we present the inverse formula of fractional Aboodh transform (1).

Theorem 2: The fractional inverse of fractional Aboodh transform (1) is defined by:

$$f(x)=\frac{1}{(\mu_\alpha)^\alpha}\int_{-i\infty}^{i\infty}p\,E_\alpha(px)^\alpha k(p)(dp)^\alpha$$

Proof:

$$f(x)=\frac{1}{(\mu_\alpha)^\alpha}\int_{-i\infty}^{i\infty}p\,E_\alpha(px)^\alpha\left(\frac{1}{p}\int_0^\infty E_\alpha(-(pu)^\alpha)f(u)(du)^\alpha\right)(dp)^\alpha$$
$$=\frac{1}{(\mu_\alpha)^\alpha}\int_0^\infty f(u)(du)^\alpha\int_{-i\infty}^{i\infty}E_\alpha\left(p^\alpha(x-u)^\alpha\right)(dp)^\alpha$$
$$=\frac{1}{\alpha}\int_0^\infty f(u)\delta_\alpha(x-u)(du)^\alpha$$
$$=\frac{1}{\alpha}.\alpha f(x)=f(x).$$

IX. Conclusion

In this present article, we define the fractional Aboodh transform and its inverse, and several theorems and properties of fractional Aboodh transform have been presented and proved, our results are consistent with Aboodh transform when $\alpha=1$.

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